

# ON THE BOUND STATES OF SCHRÖDINGER OPERATORS WITH $\delta$ -INTERACTIONS ON CONICAL SURFACES

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**ABSTRACT.** In dimension greater than or equal to three, we investigate the spectrum of a Schrödinger operator with a  $\delta$ -interaction supported on a cone whose cross section is the sphere of co-dimension two. After decomposing into fibers, we prove that there is discrete spectrum only in dimension three and that it is generated by the axisymmetric fiber. We get that these eigenvalues are non-decreasing functions of the aperture of the cone and we exhibit the precise logarithmic accumulation of the discrete spectrum below the threshold of the essential spectrum.

## 1. INTRODUCTION

**1.1. Motivation.** Some physical systems are efficiently described by Schrödinger operators with singular  $\delta$ -type interactions supported on various sets of zero Lebesgue measure (points, curves, surfaces or hypersurfaces). For instance, these operators are used to approximate atomic Hamiltonians in strong homogeneous magnetic fields [8] or photonic crystals with high-contrast [20]. The spectra of such Schrödinger operators are related to admissible values of the energy in quantum mechanics or, to admissible propagation frequencies of electromagnetic waves in optics.

A natural issue is to understand how the geometry of the support of the  $\delta$ -interaction influences the spectrum of these Schrödinger operators. This question is important, not only because of prospective applications in physics, but because it is also mathematically relevant in spectral geometry. For references on this topic, we refer to the review paper [13], the monograph [17] and the references therein.

In dimension two, for  $\delta$ -interactions supported on curves, the question of finding a connection between the spectrum and the geometry was first addressed in [14]. In that paper, the authors consider two-dimensional Schrödinger operators with attractive  $\delta$ -interactions supported on asymptotically straight curves. Provided that the curve is not a straight line, they prove that there exists at least one bound state below the threshold of the essential spectrum. In the same spirit, we mention the special case of  $\delta$ -interactions supported on broken lines, investigated in [7, 12, 16, 18].

In dimension three the state of the art is not as complete as in dimension two. Instead of dealing with asymptotically straight curves, one is interested in attractive  $\delta$ -interactions supported on asymptotically flat surfaces and such Schrödinger operators were first studied in [15]. Provided the cross section is smooth, infinite conical surfaces give rise to a family of asymptotically flat surfaces. The special case of a circular cross section is investigated in [3] where the main result is the existence of infinitely many bound states below the threshold of the essential spectrum. Moreover, the authors bound from above the sequence of eigenvalues by a sequence which converges to the threshold of the essential spectrum at a known rate. Nevertheless, sharp spectral asymptotics on the number of eigenvalues remained unknown so far. We tackle this question in the present paper and give the precise rate of accumulation. It is reminiscent of [11], where the authors exhibit a similar result for a Dirichlet Laplacian in a conical layer. In this last paper, the authors also study the behaviour of the eigenvalues with respect to the aperture of the cone. Here, we restrain ourselves to show that the eigenvalues depend monotonously on the aperture of the cone.

In dimension greater than or equal to four very little is known so far. In the special case of an attractive  $\delta$ -interaction supported on a hyperconical surface we prove that there is no discrete spectrum. Because a hyperconical surface

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splits the Euclidean space into a convex domain and a non-convex conical domain, it is worth mentioning that this result strengthens the difference with Robin Laplacians in convex circular conical domains. Indeed, according to [25], these Robin Laplacians have infinite discrete spectrum for any dimension greater than or equal to three.

Finally, we emphasise that, for attractive  $\delta$ -interactions supported on conical surfaces, the structure of the essential spectrum strongly depends on the smoothness of the cross section. For attractive  $\delta$ -interactions supported on general non-smooth conical surfaces this structure is expected to be more involved and we refer to [5] for similar considerations about magnetic Laplacians on non-smooth conical domains.

**1.2. Hamiltonians with  $\delta$ -interactions on conical surfaces.** For  $d \geq 3$ , let  $(x_1, x_2, x_3, \dots, x_d)$  be the Cartesian coordinates on  $\mathbb{R}^d$ .  $(L^2(\mathbb{R}^d); (\cdot, \cdot)_{\mathbb{R}^d})$  and  $(L^2(\mathbb{R}^d; \mathbb{C}^d); (\cdot, \cdot)_{\mathbb{R}^d})$  denote the usual  $L^2$ -space and the  $L^2$ -space of vector-valued functions over  $\mathbb{R}^d$ , respectively. The first order  $L^2$ -based Sobolev space over  $\mathbb{R}^d$  is denoted by  $H^1(\mathbb{R}^d)$ . We define the function  $\mathbb{R}^d \ni x \mapsto \rho(x) \in \mathbb{R}_+$  as

$$(1.1) \quad \rho(x) := \sqrt{\sum_{k=1}^{d-1} x_k^2}$$

and introduce the conical hypersurface  $\mathcal{C}_{d,\theta} \subset \mathbb{R}^d$ , as

$$(1.2) \quad \mathcal{C}_{d,\theta} := \{x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d : x_d = (\cot \theta) \rho(x)\}, \quad \theta \in (0, \pi/2).$$

The parameter  $\theta$  is the half-opening angle (aperture) of the cone (cf. Figure 1.1). Since there is no possible confusion, we denote the conical hypersurface by  $\mathcal{C}$  instead of  $\mathcal{C}_{d,\theta}$ . We also denote by  $(L^2(\mathcal{C}); (\cdot, \cdot)_{\mathcal{C}})$  the  $L^2$ -space over  $\mathcal{C}$ .

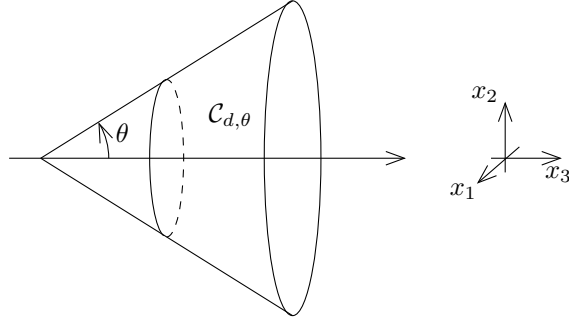


FIGURE 1.1. The cone  $\mathcal{C}_{d,\theta}$  in dimension  $d = 3$ .

For  $\alpha > 0$ , we introduce the symmetric, densely defined sesquilinear form

$$(1.3) \quad Q_{\alpha,\mathcal{C}}[u, v] := (\nabla u, \nabla v)_{\mathbb{R}^d} - \alpha(u|_{\mathcal{C}}, v|_{\mathcal{C}})_{\mathcal{C}}, \quad \text{dom } Q_{\alpha,\mathcal{C}} := H^1(\mathbb{R}^d).$$

This form is closed and semibounded in  $L^2(\mathbb{R}^d)$  (cf., e.g., [6, Sec. 2] and [2, Prop. 3.1]).

**Definition 1.1.** By the first representation theorem ([22, Ch. VI, Thm. 2.1]), the form  $Q_{\alpha,\mathcal{C}}$  is associated with a self-adjoint operator  $H_{\alpha,\mathcal{C}}$  acting on  $L^2(\mathbb{R}^d)$ . This operator is called Schrödinger operator with  $\delta$ -interaction of strength  $\alpha > 0$  supported on  $\mathcal{C}$ .

The operator  $H_{\alpha,\mathcal{C}}$  can be understood as the Laplacian on  $\mathbb{R}^d$  with a  $\delta$ -type coupling boundary condition on the conical surface  $\mathcal{C}$ . Formally, one can write  $H_{\alpha,\mathcal{C}} = -\Delta - \alpha\delta_{\mathcal{C}}$  or  $H_{\alpha,\mathcal{C}} = -\Delta - \alpha\delta(x - \mathcal{C})$ . We refer to [2], for a rigorous description of the action of  $H_{\alpha,\mathcal{C}}$  and of its domain.

**1.3. Notations and main results.** We introduce a few notation before stating the main results of this paper. The set of positive integers is denoted by  $\mathbb{N} := \{1, 2, \dots\}$  and the set of natural integers is denoted by  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . Let  $T$  be a semi-bounded self-adjoint operator associated with the quadratic form  $t$ . We denote by  $\sigma_{\text{ess}}(T)$  and  $\sigma_{\text{dis}}(T)$  the essential and the discrete spectrum of  $T$ , respectively. By  $\sigma(T)$ , we denote the spectrum of  $T$  (i.e.  $\sigma(T) = \sigma_{\text{ess}}(T) \cup \sigma_{\text{dis}}(T)$ ).

We set  $E_{\text{ess}}(\mathsf{T}) := \inf \sigma_{\text{ess}}(\mathsf{T})$  and, for  $k \in \mathbb{N}$ ,  $E_k(\mathsf{T})$  denotes the  $k$ -th eigenvalue of  $\mathsf{T}$  in the interval  $(-\infty, E_{\text{ess}}(\mathsf{T}))$ . They are ordered non-decreasingly with multiplicities taken into account. We define the counting function of  $\mathsf{T}$  as

$$\mathcal{N}_E(\mathsf{T}) := \#\{k \in \mathbb{N}: E_k(\mathsf{T}) < E\}, \quad E \leq E_{\text{ess}}(\mathsf{T}).$$

When working with the quadratic form  $\mathfrak{t}$ , we use the notations  $\sigma_{\text{ess}}(\mathfrak{t})$ ,  $\sigma_{\text{dis}}(\mathfrak{t})$ ,  $\sigma(\mathfrak{t})$ ,  $E_{\text{ess}}(\mathfrak{t})$ ,  $E_k(\mathfrak{t})$  and  $\mathcal{N}_E(\mathfrak{t})$  instead.

The first result is about the characterisation of the essential spectrum and the qualitative description of the discrete spectrum for  $H_{\alpha, \mathcal{C}}$ .

**Theorem 1.2.** *Let  $\theta \in (0, \pi/2)$  and  $\alpha > 0$ . The following statements hold:*

- (i) *for any dimension  $d \geq 3$ ,  $\sigma_{\text{ess}}(H_{\alpha, \mathcal{C}}) = [-\alpha^2/4, +\infty)$ ;*
- (ii) *for  $d = 3$ ,  $\#\sigma_{\text{dis}}(H_{\alpha, \mathcal{C}}) = \infty$ ;*
- (iii) *for  $d \geq 4$ ,  $\#\sigma_{\text{dis}}(H_{\alpha, \mathcal{C}}) = 0$ .*

Note that Theorem 1.2 was already known in dimension  $d = 3$  (cf. [3]). As the proof of the structure of the essential spectrum in dimension  $d \geq 4$  follows exactly the same lines as the one exposed in [3, §2] (in dimension  $d = 3$ ), we omit it here for the sake of brevity. The absence of discrete spectrum in item (iii) differs from the results in [14, 15] where, in dimension  $d = 2$  or  $d = 3$ , it is shown that geometric deformations always induce bound states (at least for  $\alpha > 0$  sufficiently large). To prove item (iii), we show that the operator  $H_{\alpha, \mathcal{C}}$  is unitarily equivalent to an infinite orthogonal sum of self-adjoint fiber operators and that the spectrum of each fiber operator is included in  $[-\alpha^2/4, +\infty)$ .

Then, relying on Theorem 1.2, we focus on properties of the discrete spectrum of  $H_{\alpha, \mathcal{C}}$  in dimension  $d = 3$ .

**Proposition 1.3.** *Let  $\alpha > 0$ . In dimension  $d = 3$ , for all  $k \in \mathbb{N}$  the functions,  $\theta \mapsto E_k(H_{\alpha, \mathcal{C}})$  are non-decreasing on  $(0, \pi/2)$ .*

This proposition is reminiscent of a similar result in [18] for  $\delta$ -interactions supported on broken lines. Nevertheless, our proof is somewhat simpler since we do not use the Birman-Schwinger principle. The idea of the proof is to show that the discrete spectrum of  $H_{\alpha, \mathcal{C}}$ , below the point  $-\alpha^2/4$ , coincides with the discrete spectrum of the lowest fiber operator below the same point. Then, one can show that the lowest fiber operator is unitarily equivalent to another operator, whose form domain is independent of  $\theta$  and whose Rayleigh quotient is a monotone function of  $\theta$ .

Finally, we state our main result on the spectral asymptotics of  $H_{\alpha, \mathcal{C}}$  for  $d = 3$ .

**Theorem 1.4.** *Let  $\theta \in (0, \pi/2)$  and  $\alpha > 0$ . In dimension  $d = 3$ , we have*

$$\mathcal{N}_{-\alpha^2/4-E}(H_{\alpha, \mathcal{C}}) \sim \frac{\cot \theta}{4\pi} |\ln(E)|, \quad E \rightarrow 0+.$$

The proof of Theorem 1.4 is inspired by a similar strategy developed in [11] for Dirichlet conical layers. Loosely speaking, we reduce the spectral asymptotics of  $H_{\alpha, \mathcal{C}}$  to the known spectral asymptotics of the one-dimensional Schrödinger operator

$$-\frac{d^2}{dx^2} - \frac{1}{4 \sin^2 \theta} \frac{1}{x^2}, \quad \text{on } (1, +\infty),$$

with Dirichlet or Neumann boundary condition at  $x = 1$ . To this end, we use Dirichlet and Neumann bracketing combined with an IMS formula (cf. [9]). We emphasise that the geometry of the bracketings depends on the spectral parameter in a more sophisticated way than in [11]. Moreover, to estimate the operators involved in the Dirichlet and Neumann bracketings, we need spectral properties of two specific one-dimensional Schrödinger operators with a  $\delta$ -point interaction.

**1.4. Organisation of the paper.** In Section 2, we reduce the study of  $H_{\alpha, \mathcal{C}}$  to a family of two-dimensional operators (fibers). This reduction allows to understand the structure of the discrete spectrum mentioned in Theorem 1.2 and to prove Proposition 1.3. After introducing one-dimensional model operators, Theorem 1.4 is proven in Section 3. Finally, we conclude the paper with Appendix A about some properties of fiber decompositions.

## 2. FIBER DECOMPOSITION AND ITS FIRST APPLICATIONS

In this section we prove Theorem 1.2 and Proposition 1.3. To take advantage of the symmetry of the problem, we work with a description of  $H_{\alpha, \mathcal{C}}$  in cylindrical coordinates. Then, we reduce the study to a family of two-dimensional operators (fibers). For  $d = 3$ , the fiber decomposition is also used in the proof of the spectral asymptotics in Section 3.

**2.1. Hyper-cylindrical coordinates.** Since the conical surface  $\mathcal{C}$  is axisymmetric, the problem is better described in (hyper-)cylindrical coordinates with  $x_d$  as the reference axis. Let us denote these coordinates by  $(r, z, \phi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{S}^{d-2}$ , where  $\mathbb{S}^{d-2} \cong [0, \pi]^{d-3} \times [0, 2\pi]$  is the unit sphere of dimension  $d-2$  and  $\mathbf{m}_{d-2}$  its natural surface measure. We can write  $\phi \in \mathbb{S}^{d-2}$  as  $\phi = (\phi_1, \dots, \phi_{d-2}) \in [0, \pi]^{d-3} \times [0, 2\pi]$  and, for all  $k \in \{1, \dots, d-2\}$ , we have

$$(2.1) \quad x_k = r \left( \prod_{p=1}^{k-1} \sin \phi_p \right) \cos \phi_k, \quad x_{d-1} = r \prod_{p=1}^{d-2} \sin \phi_p, \quad x_d = z.$$

For further use, we introduce the *meridian domain*  $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}$ , the *meridian ray*

$$(2.2) \quad \Gamma_\theta := \{(r, z) \in \mathbb{R}_+^2 : z = r \cot \theta\},$$

Now, we introduce some notations related to the cylindrical coordinates.

**Notation 2.1.**  $L_{\text{cyl}}^2(\mathbb{R}^d)$  denotes the Hilbert space

$$L_{\text{cyl}}^2(\mathbb{R}^d) = L^2(\mathbb{R}_+^2 \times \mathbb{S}^{d-2}, r^{d-2} dr dz d\mathbf{m}_{d-2}(\phi)).$$

Similarly, we define the Sobolev cylindrical space  $H_{\text{cyl}}^1(\mathbb{R}^d)$  by

$$H_{\text{cyl}}^1(\mathbb{R}^d) := \{u \in L_{\text{cyl}}^2(\mathbb{R}^d) : \partial_r u, \partial_z u, r^{-1} |\nabla_{\mathbb{S}^{d-2}} u| \in L_{\text{cyl}}^2(\mathbb{R}^d)\},$$

endowed with the norm

$$\|u\|_{H_{\text{cyl}}^1(\mathbb{R}^d)}^2 := \|u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 + \|\partial_r u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 + \|\partial_z u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 + \|r^{-1} |\nabla_{\mathbb{S}^{d-2}} u|\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2,$$

where  $\nabla_{\mathbb{S}^{d-2}}$  is the surface gradient on  $\mathbb{S}^{d-2}$ .

$\mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2})$  denotes the space of infinitely differentiable functions with compact support in  $\overline{\mathbb{R}_+^2}$  and we introduce the space  $\mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$  defined by

$$\mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2}) = \{u \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2}) : u|_{r=0} = 0\}.$$

The change of variables (2.1) maps the whole space  $\mathbb{R}^d$  onto  $\mathbb{R}_+^2 \times \mathbb{S}^{d-2}$  and  $H_{\alpha, \mathcal{C}}$  becomes the unbounded self-adjoint operator  $L_{\alpha, \mathcal{C}}$  acting on  $L_{\text{cyl}}^2(\mathbb{R}^d)$ . By the first representation theorem,  $L_{\alpha, \mathcal{C}}$  can be seen as the operator associated with the quadratic form  $Q_{\alpha, \mathcal{C}}^{\text{cyl}}$ , defined by the expression of  $Q_{\alpha, \mathcal{C}}$  in cylindrical coordinates:

$$\begin{aligned} Q_{\alpha, \mathcal{C}}^{\text{cyl}}[u] &:= \|\partial_r u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 + \|\partial_z u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 + \|r^{-1} |\nabla_{\mathbb{S}^{d-2}} u|\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 \\ &\quad - \alpha \int_{\mathbb{R}_+ \times \mathbb{S}^{d-2}} |u(s \sin \theta, s \cos \theta, \phi)|^2 ds d\mathbf{m}_{d-2}(\phi). \end{aligned}$$

$$\text{dom } Q_{\alpha, \mathcal{C}}^{\text{cyl}} := H_{\text{cyl}}^1(\mathbb{R}^d).$$

**2.2. Spherical harmonics.** First, we recall some known results on the spherical harmonics that can be found, *e.g.*, in [28, Chap. IV §2].  $-\Delta_{\mathbb{S}^{d-2}}$  denotes the Laplace-Beltrami on the sphere  $\mathbb{S}^{d-2}$  and its eigenvalues are given by  $l(l+d-3)$ , with  $l \in \mathbb{N}_0$  and the associated eigenspaces, denoted by  $\mathcal{G}_l^{d-2}$ , are of dimension

$$c(d, l) := \binom{d+l-2}{d-2} - \binom{d+l-4}{d-2}, \quad \text{where} \quad \binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } n \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for all  $k \in \{1, \dots, c(d, l)\}$ , we denote by  $Y_{l,k}^{d-2}$  the usual spherical harmonics. For all  $l \in \mathbb{N}_0$  and  $k \in \{1, \dots, c(d, l)\}$ , they satisfy

$$-\Delta_{\mathbb{S}^{d-2}} Y_{l,k}^{d-2} = l(l+d-3) Y_{l,k}^{d-2}, \quad \mathcal{G}_l^{d-2} = \text{span}_{k \in \{1, \dots, c(d, l)\}} \{Y_{l,k}^{d-2}\}.$$

Finally, the family of all the spherical harmonics forms an orthonormal basis of  $L^2(\mathbb{S}^{d-2})$ ; *i.e.*,

$$\overline{\text{span}_{l \in \mathbb{N}_0} \mathcal{G}_l^{d-2}} = L^2(\mathbb{S}^{d-2}).$$

Now, decomposing into spherical harmonics and according to the terminology of [26, §XIII.16], we have the constant fiber sum

$$\begin{aligned} L^2(\mathbb{R}_+^2 \times \mathbb{S}^{d-2}, r^{d-2} dr dz d\mathbf{m}_{d-2}(\phi)) &= L^2(\mathbb{R}_+^2, r^{d-2} dr dz) \otimes L^2(\mathbb{S}^{d-2}) \\ &= \bigoplus_{l \in \mathbb{N}_0} \bigoplus_{k=1}^{c(d, l)} L^2(\mathbb{R}_+^2, r^{d-2} dr dz). \end{aligned}$$

Secondly, for any function  $u \in L_{\text{cyl}}^2(\mathbb{R}^d)$ , we define

$$(\pi_{l,k} u)(r, z) := \left\langle u, Y_{l,k}^{d-2} \right\rangle_{\mathbb{S}^{d-2}} = \int_{\mathbb{S}^{d-2}} u(r, z, \phi) \overline{Y_{l,k}^{d-2}(\phi)} d\mathbf{m}_{d-2}(\phi).$$

We consider the family  $(\Pi_{l,k})_{l \in \mathbb{N}_0, k \in \{1, \dots, c(d, l)\}}$  of orthogonal projectors on  $L_{\text{cyl}}^2(\mathbb{R}^d)$ , defined as

$$(2.3) \quad (\Pi_{l,k} u)(r, z, \phi) := (\pi_{l,k} u)(r, z) Y_{l,k}^{d-2}(\phi).$$

By definition of the spherical harmonics and  $H_{\text{cyl}}^1(\mathbb{R}^d)$ , we know that  $\Pi_{l,k}(H_{\text{cyl}}^1(\mathbb{R}^d)) \subset H_{\text{cyl}}^1(\mathbb{R}^d)$ . We introduce the quadratic forms

$$(2.4) \quad Q_{\alpha, \Gamma_\theta}^{[l, k]}[u] := Q_{\alpha, C}^{\text{cyl}}[u Y_{l,k}^{d-2}], \quad \text{dom } Q_{\alpha, \Gamma_\theta}^{[l, k]} := \pi_{l,k}(H_{\text{cyl}}^1(\mathbb{R}^d)).$$

By straightforward computations we first notice that the forms  $Q_{\alpha, \Gamma_\theta}^{[l, k]}$  do not depend on  $k$  and to simplify, we drop the index  $k$ . Thirdly, we get

$$\begin{aligned} (2.5) \quad Q_{\alpha, \Gamma_\theta}^{[l]}[u] &= \int_{\mathbb{R}_+^2} \left( |\partial_r u|^2 + |\partial_z u|^2 + \frac{l(l+d-3)}{r^2} |u|^2 \right) r^{d-2} dr dz \\ &\quad - \alpha \int_{\mathbb{R}_+} |u(s \sin \theta, s \cos \theta)|^2 s^{d-2} \sin^{d-2} \theta ds, \\ \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]} &= \begin{cases} \{u : u, \partial_r u, \partial_z u \in L^2(\mathbb{R}_+^2, r^{d-2} dr dz)\}, & l = 0, \\ \{u : u, \partial_r u, \partial_z u, r^{-1} u \in L^2(\mathbb{R}_+^2, r^{d-2} dr dz)\}, & l \neq 0. \end{cases} \end{aligned}$$

We refer to [4, §II.3.a] for a full description of the domains of the above forms when  $d = 3$ .

Using (2.4), one can show that the quadratic forms  $Q_{\alpha, \Gamma_\theta}^{[l]}$  are symmetric, closed, densely defined and semibounded on  $\pi_{l,k}(L_{\text{cyl}}^2(\mathbb{R}^d)) = L^2(\mathbb{R}_+^2, r^{d-2} dr dz)$ . Hence, by the first representation theorem, each quadratic form  $Q_{\alpha, \Gamma_\theta}^{[l]}$  is associated with a self-adjoint operator  $\mathcal{L}_{\alpha, \Gamma_\theta}^{[l]}$  acting on  $L^2(\mathbb{R}_+^2, r^{d-2} dr dz)$ .

Using the precise description of  $\text{dom } \mathcal{L}_{\alpha, C}$  given in [2, Thm. 3.3 (a)] and the symmetry of  $\mathcal{C}$ , one can show that

$$\Pi_{l,k}(\text{dom } \mathcal{L}_{\alpha, C}) \subset \text{dom } \mathcal{L}_{\alpha, C}, \quad \mathcal{L}_{\alpha, C}(\Pi_{l,k}(\text{dom } \mathcal{L}_{\alpha, C})) \subset \Pi_{l,k}(L_{\text{cyl}}^2(\mathbb{R}^d)).$$

The first representation theorem implies that the operator  $\pi_{l,k} \mathsf{L}_{\alpha,C} |_{\Pi_{l,k}(\text{dom } \mathsf{L}_{\alpha,C})}$  can be identified with  $\mathsf{L}_{\alpha,\Gamma_\theta}^{[l]}$  and that they have the same spectrum. By [27, §1.4], the operator  $\mathsf{L}_{\alpha,C}$  decomposes as

$$(2.6) \quad \mathsf{L}_{\alpha,C} = \bigoplus_{l \in \mathbb{N}_0} \bigoplus_{k=1}^{c(d,l)} \mathsf{L}_{\alpha,\Gamma_\theta}^{[l]}.$$

The self-adjoint operators  $\mathsf{L}_{\alpha,\Gamma_\theta}^{[l]}$  are the fibers of  $\mathsf{L}_{\alpha,C}$  and this decomposition yields

$$(2.7) \quad \sigma(\mathsf{H}_{\alpha,C}) = \sigma(\mathsf{L}_{\alpha,C}) = \overline{\bigcup_{l \in \mathbb{N}_0} \bigcup_{k=1}^{c(d,l)} \sigma(\mathsf{L}_{\alpha,\Gamma_\theta}^{[l]})} = \overline{\bigcup_{l \in \mathbb{N}_0} \sigma(\mathsf{L}_{\alpha,\Gamma_\theta}^{[l]})}.$$

For further use, for all  $d \geq 3$  and  $l \geq 0$ , we define  $\text{Co}(Q_{\alpha,\Gamma_\theta}^{[l]})$  as

$$\text{Co}(Q_{\alpha,\Gamma_\theta}^{[l]}) := \begin{cases} \mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2}), & \text{when } (d,l) = (3,l), \quad l > 0, \\ \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2}), & \text{otherwise.} \end{cases}$$

With this definition,  $\text{Co}(Q_{\alpha,\Gamma_\theta}^{[l]})$  is a form core of  $Q_{\alpha,\Gamma_\theta}^{[l]}$  (cf. Proposition A.1).

**2.3. Flat metric.** In this subsection, after reformulating the problem in flat metric, we study the quadratic forms  $Q_{\alpha,\Gamma_\theta}^{[l]}$  with the help of a unitarily equivalent form. First, we formulate the following proposition.

**Proposition 2.2.** *Let  $\mathcal{H}$  and  $\mathcal{G}$  be two Hilbert spaces and let  $U: \mathcal{H} \rightarrow \mathcal{G}$  be a unitary operator. Let  $\mathfrak{s}$  be a closed, densely defined, symmetric and semibounded quadratic form on the Hilbert space  $\mathcal{H}$ . Define the quadratic form  $\mathfrak{t}$  by*

$$\mathfrak{t}[u] := \mathfrak{s}[U^{-1}v], \quad \text{dom } \mathfrak{t} := U(\text{dom } \mathfrak{s}).$$

*With this definition, the following statements hold.*

- (i) *The form  $\mathfrak{t}$  is closed, densely defined, symmetric, and semibounded on the Hilbert space  $\mathcal{G}$ ; and we say that  $\mathfrak{s}$  and  $\mathfrak{t}$  are unitarily equivalent.*
- (ii) *The respective self-adjoint operators  $T$  and  $S$  (associated with these forms) are unitarily equivalent and the relation  $S = U^{-1}TU$  holds.*
- (iii) *If  $\text{Co}(\mathfrak{s})$  is a form core of  $\mathfrak{s}$ , then  $U(\text{Co}(\mathfrak{s}))$  is a form core of  $\mathfrak{t}$ .*

Proposition 2.2 is a direct consequence of the definition of  $\mathfrak{t}$  and the definition of the unitary transform.

Secondly, we introduce the following unitary transform

$$(2.8) \quad U: L^2(\mathbb{R}_+^2, r^{d-2} dr dz) \rightarrow L^2(\mathbb{R}_+^2), \quad (Uu)(r, z) := r^{(d-2)/2} u(r, z) \equiv \tilde{u}(r, z).$$

**Proposition 2.3.** *Let  $d \geq 3$  and  $l \in \mathbb{N}_0$  be such that  $(d, l) \neq (3, 0)$ . Then  $Q_{\alpha,\Gamma_\theta}^{[l]}$  is unitarily equivalent to the quadratic form*

$$\begin{aligned} \tilde{Q}_{\alpha,\Gamma_\theta}^{[l]}[\tilde{u}] &:= \int_{\mathbb{R}_+^2} \left( |\partial_r \tilde{u}|^2 + |\partial_z \tilde{u}|^2 + \frac{\gamma(d,l)}{r^2} |\tilde{u}|^2 \right) dr dz - \alpha \int_{\mathbb{R}_+} |\tilde{u}(s \sin \theta, s \cos \theta)|^2 ds, \\ \text{dom } \tilde{Q}_{\alpha,\Gamma_\theta}^{[l]} &:= U(\text{dom } Q_{\alpha,\Gamma_\theta}^{[l]}), \end{aligned}$$

where  $\gamma(d, l) = l(l + d - 3) + \frac{1}{4}(d - 2)(d - 4) \geq 0$ . Moreover,  $U(\text{Co}(Q_{\alpha,\Gamma_\theta}^{[l]}))$  is a form core of  $\tilde{Q}_{\alpha,\Gamma_\theta}^{[l]}$  and any function  $\tilde{u} \in U(\text{Co}(Q_{\alpha,\Gamma_\theta}^{[l]}))$  satisfies  $\tilde{u}|_{r=0} = 0$ .

*Proof.* Let  $(d, l) \neq (3, 0)$ . First, we define the quadratic form  $\tilde{Q}_{\alpha,\Gamma_\theta}^{[l]}$  as

$$\tilde{Q}_{\alpha,\Gamma_\theta}^{[l]}[\tilde{u}] := Q_{\alpha,\Gamma_\theta}^{[l]}[U^{-1}\tilde{u}], \quad \text{dom } \tilde{Q}_{\alpha,\Gamma_\theta}^{[l]} := U(\text{dom } Q_{\alpha,\Gamma_\theta}^{[l]}),$$

with  $U$  as in (2.8). By Proposition 2.2(i) we know that  $\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}$  is a closed, densely defined, symmetric, and semi-bounded quadratic form on  $L^2(\mathbb{R}_+^2)$ . To get the expression of  $\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}$  stated in Proposition 2.3, it is sufficient to check it on a form core of  $\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}$ . Consequently, let us choose  $u \in \text{Co}(Q_{\alpha, \Gamma_\theta}^{[l]})$  and set  $\tilde{u} := Uu$ . Then, we have

$$\begin{aligned} \tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}[\tilde{u}] &= Q_{\alpha, \Gamma_\theta}^{[l]}[r^{-(d-2)/2}\tilde{u}] = \int_{\mathbb{R}_+^2} |\partial_r(r^{-(d-2)/2}\tilde{u})|^2 r^{d-2} dr dz + \int_{\mathbb{R}_+^2} \left( |\partial_z \tilde{u}|^2 + \frac{l(l+d-3)}{r^2} |\tilde{u}|^2 \right) dr dz \\ &\quad - \alpha \int_{\mathbb{R}_+} |\tilde{u}(s \sin \theta, s \cos \theta)|^2 ds. \end{aligned}$$

A simple computation yields

$$(2.9) \quad |\partial_r u|^2 = |\partial_r(r^{-(d-2)/2}\tilde{u})|^2 = \frac{1}{r^{d-2}} |\partial_r \tilde{u}|^2 + \frac{(d-2)^2}{4r^d} |\tilde{u}|^2 - \frac{d-2}{2r^{d-1}} \partial_r(|\tilde{u}|^2).$$

Integrating over  $r \in \mathbb{R}_+$  the last term in (2.9), and integrating by parts, we end up with

$$(2.10) \quad \int_{\mathbb{R}_+} \frac{1}{r^{d-1}} \partial_r(|\tilde{u}|^2) r^{d-2} dr = \int_{\mathbb{R}_+} \frac{1}{r} \partial_r(|\tilde{u}|^2) dr = \lim_{r \rightarrow +\infty} (r^{-1} |\tilde{u}|^2) - \lim_{r \rightarrow 0} (r^{-1} |\tilde{u}|^2) + \int_{\mathbb{R}_+} \frac{1}{r^2} |\tilde{u}|^2 dr.$$

The two limits in (2.10) make sense and both are equal to zero. Indeed, we have  $r^{-1} |\tilde{u}|^2 = |u|^2 r^{d-3}$ , and  $u$  is compactly supported, so  $|\tilde{u}|^2 = |u|^2 r^{d-2} \rightarrow 0$  as  $r \rightarrow +\infty$  and the first limit in (2.10) is zero. Now, when  $d = 3$ ,  $u \in C_{0,0}^\infty(\mathbb{R}_+^2)$  so,  $r^{-1} |\tilde{u}|^2 = |u|^2 \rightarrow 0$  as  $r \rightarrow 0$ . When  $d \geq 4$ ,  $u \in C_0^\infty(\overline{\mathbb{R}_+^2})$  and  $r^{-1} |\tilde{u}|^2 = |u|^2 r^{d-3} \rightarrow 0$  as  $r \rightarrow 0$ . Hence, (2.10) rewrites as

$$(2.11) \quad \int_{\mathbb{R}_+} \frac{1}{r^{d-1}} \partial_r(|\tilde{u}|^2) r^{d-2} dr = \int_{\mathbb{R}_+} \frac{1}{r^2} |\tilde{u}|^2 dr.$$

Using (2.9) and (2.11) we get the desired expression for  $\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}$ .  $\square$

Now, we are ready to prove the following statement.

**Proposition 2.4.** *Let  $d \geq 3$  and  $l \in \mathbb{N}_0$  be such that  $(d, l) \neq (3, 0)$ . Then we have*

$$\inf \sigma(L_{\alpha, \Gamma_\theta}^{[l]}) \geq -\alpha^2/4.$$

*Proof.* Instead of working with the operator  $L_{\alpha, \Gamma_\theta}^{[l]}$  and its associated form  $Q_{\alpha, \Gamma_\theta}^{[l]}$  we work with the unitarily equivalent quadratic form in the flat metric  $\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}$ . We want to apply the min-max principle to the quadratic form  $\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}$ . To do so, it is sufficient to apply it with test functions in  $U(\text{Co}(Q_{\alpha, \Gamma_\theta}^{[l]}))$ , where  $U$  is as in (2.8). Let  $\tilde{u} \in U(\text{Co}(Q_{\alpha, \Gamma_\theta}^{[l]}))$ , we have

$$\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}[\tilde{u}] \geq \int_{\mathbb{R}_+^2} (|\partial_r \tilde{u}|^2 + |\partial_z \tilde{u}|^2) dr dz - \alpha \int_{\mathbb{R}_+} |\tilde{u}(s \sin \theta, s \cos \theta)|^2 ds.$$

Thanks to Proposition 2.3 we know that  $\tilde{u} \in H_0^1(\mathbb{R}_+^2)$ .  $\tilde{u}$  can be extended by zero to the whole plane  $\mathbb{R}^2$ , defining a function  $\tilde{u}_0 \in H^1(\mathbb{R}^2)$ . We obtain

$$(2.12) \quad \tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}[\tilde{u}] \geq \int_{\mathbb{R}^2} (|\partial_r \tilde{u}_0|^2 + |\partial_z \tilde{u}_0|^2) dr dz - \alpha \int_{\mathbb{R}} |\tilde{u}_0(s \sin \theta, s \cos \theta)|^2 ds.$$

The quadratic form on the right-hand side is the one of a Schrödinger operator with an attractive  $\delta$ -interaction of strength  $\alpha > 0$  supported on a straight line in  $\mathbb{R}^2$ . Its spectrum can be computed *via* separation of variables and is  $[-\alpha^2/4, +\infty)$ . The min-max principle applied to the form on the right hand side of (2.12) yields

$$\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}[\tilde{u}] \geq -(\alpha^2/4) \|\tilde{u}_0\|_{\mathbb{R}^2}^2 = -(\alpha^2/4) \|\tilde{u}\|_{\mathbb{R}_+^2}^2.$$

Finally, we get the inequality applying the min-max principle to  $\tilde{Q}_{\alpha, \Gamma_\theta}^{[l]}$ .  $\square$

Combining the structure of the essential spectrum, stated in Theorem 1.2, with Proposition 2.4 we obtain the following corollary.

**Corollary 2.5.** *Let the self-adjoint operator  $H_{\alpha,C}$  be as in Definition 1.1 and the self-adjoint operator  $L_{\alpha,\Gamma_\theta}^{[0]}$  be as in (2.6).*

- (i) *For  $d = 3$ ,  $\sigma_{\text{dis}}(H_{\alpha,C}) = \sigma_{\text{dis}}(L_{\alpha,\Gamma_\theta}^{[0]})$  holds and the multiplicities of the corresponding eigenvalues coincide.*
- (ii) *For  $d \geq 4$ ,  $\sigma_{\text{dis}}(H_{\alpha,C}) = \emptyset$  holds.*

In dimension  $d \geq 4$ , it proves Theorem 1.2(iii) about the emptiness of  $\sigma_{\text{dis}}(H_{\alpha,C})$ . In dimension  $d = 3$ , it reduces the study of the eigenvalues of  $L_{\alpha,C}$  to its axisymmetric fiber  $L_{\alpha,\Gamma_\theta}^{[0]}$ . In the remainder of this paper, except if stated explicitly,  $d = 3$  and to simplify the notations, we drop the index 0 and define

$$(2.13) \quad L_{\alpha,\Gamma_\theta} := L_{\alpha,\Gamma_\theta}^{[0]}, \quad Q_{\alpha,\Gamma_\theta} := Q_{\alpha,\Gamma_\theta}^{[0]}.$$

**2.4. Monotonicity of the eigenvalues.** We prove Proposition 1.3 about the monotonicity of the eigenvalues of  $H_{\alpha,C}$  with respect to the half-opening angle of the underlying cone  $\mathcal{C}$ . Thanks to Corollary 2.5, we know that we only have to focus on the axisymmetric fiber  $L_{\alpha,\Gamma_\theta}$ . We describe the transition from the fiber form  $Q_{\alpha,\Gamma_\theta}$  (in (2.13)) on the meridian domain to a unitarily equivalent form on the inclined half-plane. This transition will be useful in the proof of Proposition 1.3 as well as in further considerations.

To this end, first, we define the rotation

$$(2.14) \quad s = z \cos \theta + r \sin \theta, \quad t = -z \sin \theta + r \cos \theta,$$

that transforms the meridian domain  $\mathbb{R}_+^2$  into the inclined half-plane (cf. Figure 2.1)

$$(2.15) \quad \Omega_\theta := \{(s, t) \in \mathbb{R}^2 : s \sin \theta + t \cos \theta > 0\}.$$

The meridian ray  $\Gamma_\theta$ , defined in (2.2), becomes the ray

$$(2.16) \quad \Gamma := \{(s, 0) \in \Omega_\theta : s > 0\}.$$

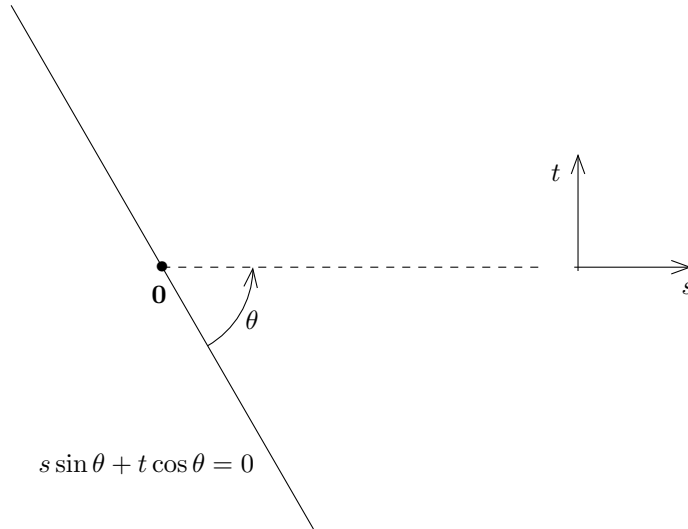


FIGURE 2.1. The inclined half-plane  $\Omega_\theta$ . The dashed line is the support of the  $\delta$ -interaction.

We associate the unitary operator

$$(2.17) \quad U: L^2(\mathbb{R}_+^2) \rightarrow L^2(\Omega_\theta), \quad (Uu)(s, t) := \widehat{u}(s, t) = u(s \sin \theta + t \cos \theta, s \cos \theta - t \sin \theta)$$



with Rotation (2.14). A straightforward computation yields that the quadratic form  $Q_{\alpha, \Gamma_\theta}$  defined in (2.13), is unitarily equivalent to  $Q_{\alpha, \Gamma}$ , defined by

$$(2.18) \quad \begin{aligned} Q_{\alpha, \Gamma}[\widehat{u}] &:= \int_{\Omega_\theta} (|\partial_s \widehat{u}|^2 + |\partial_t \widehat{u}|^2)(s \sin \theta + t \cos \theta) ds dt - \alpha \int_{\Gamma} |\widehat{u}(s, 0)|^2 s \sin \theta ds, \\ \text{dom } Q_{\alpha, \Gamma} &:= \text{U}(\text{dom } Q_{\alpha, \Gamma_\theta}). \end{aligned}$$

To avoid the dependence on  $\theta$  of the domain  $\text{dom } Q_{\alpha, \Gamma}$ , we perform the change of variables  $(s, t) \mapsto (\check{s}, \check{t}) = (s \tan \theta, t)$  that transforms the domain  $\Omega_\theta$  into  $\Omega := \Omega_{\pi/4}$ . Setting  $\check{u}(\check{s}, \check{t}) = \widehat{u}(s, t)$ , for  $u \in \text{dom } Q_{\alpha, \Gamma_\theta}$  we get, for the Rayleigh quotient

$$\begin{aligned} \frac{Q_{\alpha, \Gamma_\theta}[u]}{\|u\|_{\mathbb{R}^2_+}^2} &= \frac{\int_{\Omega} (\tan^2 \theta |\partial_{\check{s}} \check{u}|^2 + |\partial_{\check{t}} \check{u}|^2)(\check{s} + \check{t}) \cos \theta \cot \theta d\check{s} d\check{t} - \alpha \int_{\check{s} > 0} |\check{u}(\check{s}, 0)|^2 \check{s} \sin \theta \cot^2 \theta d\check{s}}{\int_{\Omega} |\check{u}|^2 (\check{s} + \check{t}) \cos \theta \cot \theta d\check{s} d\check{t}} \\ &= \frac{\int_{\Omega} (\tan^2 \theta |\partial_{\check{s}} \check{u}|^2 + |\partial_{\check{t}} \check{u}|^2)(\check{s} + \check{t}) d\check{s} d\check{t} - \alpha \int_{\check{s} > 0} |\check{u}(\check{s}, 0)|^2 \check{s} d\check{s}}{\int_{\Omega} |\check{u}|^2 (\check{s} + \check{t}) d\check{s} d\check{t}}. \end{aligned}$$

The right hand side of the last equation is the Rayleigh quotient of a quadratic form acting on  $L^2(\Omega; (\check{s} + \check{t}) d\check{s} d\check{t})$ . Because this form is unitarily equivalent to  $Q_{\alpha, \Gamma_\theta}$  and because its Rayleigh quotients are nondecreasing functions of  $\theta$ , the claim of Proposition 1.3 follows from the min-max formulae for the eigenvalues.

### 3. SPECTRAL ASYMPTOTICS OF $H_{\alpha, \mathcal{C}}$ IN THE THREE-DIMENSIONAL CASE

In this section we prove Theorem 1.4. The idea is to exhibit a lower and an upper bound for the counting function of the operator  $L_{\alpha, \Gamma_\theta}$ . We recall that all along this section  $d = 3$ .

**3.1. Proof of Theorem 1.4.** The proof of Theorem 1.4 relies on the following two propositions, whose proofs are postponed to Subsections 3.3 and 3.4, respectively.

**Proposition 3.1.** *Let  $\alpha > 0$  and  $\theta \in (0, \pi/2)$ . We have*

$$\liminf_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(H_{\alpha, \mathcal{C}})}{|\ln E|} \geq \frac{\cot \theta}{4\pi}.$$

**Proposition 3.2.** *Let  $\alpha > 0$  and  $\theta \in (0, \pi/2)$ . We have*

$$\limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(H_{\alpha, \mathcal{C}})}{|\ln E|} \leq \frac{\cot \theta}{4\pi}.$$

Propositions 3.1 and 3.2 imply

$$\lim_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(H_{\alpha, \mathcal{C}})}{|\ln E|} = \frac{\cot \theta}{4\pi},$$

which proves Theorem 1.4.

**3.2. Auxiliary one-dimensional operators.** In this subsection we discuss some spectral properties of one-dimensional model Schrödinger operators, which are used in the proofs of Propositions 3.1 and 3.2.

Let us start by studying two Schrödinger operators with a point  $\delta$ -interaction. For the spectral theory of one-dimensional Schrödinger operators with point  $\delta$ -interactions we refer to [1, Chs. I.3, II.2, III.2], the review paper [24] and the references therein.

For  $L > 0$ , we define the interval  $I := (-L, L)$  and introduce the Hilbert space  $(L^2(I), (\cdot, \cdot)_I)$ . Let us fix  $\alpha > 0$  and consider the following two symmetric sesquilinear forms

$$\begin{aligned} \mathfrak{q}_{\alpha, L}^D[\varphi, \psi] &:= (\varphi', \psi')_I - \alpha \varphi(0) \overline{\psi(0)}, & \text{dom } \mathfrak{q}_{\alpha, L}^D &:= H_0^1(I), \\ \mathfrak{q}_{\alpha, L}^N[\varphi, \psi] &:= (\varphi', \psi')_I - \alpha \varphi(0) \overline{\psi(0)}, & \text{dom } \mathfrak{q}_{\alpha, L}^N &:= H^1(I), \end{aligned}$$

one can verify that both forms are closed, densely defined, and semibounded in  $L^2(I)$ . For  $\varphi \in H^2(I \setminus \{0\})$  set  $\varphi_+ := \varphi|_{(0,L)}$ ,  $\varphi_- := \varphi|_{(-L,0)}$  and  $[\varphi'](0) := \varphi'(0+) - \varphi'(0-)$ . Thanks to the first representation theorem ([22, Ch. VI, Thm. 2.1]), each of these quadratic forms is associated with a unique self-adjoint operator acting on  $L^2(I)$ . The respective operators are given by

$$(3.1) \quad \begin{aligned} h_{\alpha,L}^D \varphi &= -(\varphi_+'' \oplus \varphi_-''), \quad \text{dom } h_{\alpha,L}^D = \{\varphi \in H^2(I \setminus \{0\}) : \varphi(\pm L) = 0, [\varphi'](0) = -\alpha\varphi(0\pm)\}, \\ h_{\alpha,L}^N \varphi &= -(\varphi_+'' \oplus \varphi_-''), \quad \text{dom } h_{\alpha,L}^N = \{\varphi \in H^2(I \setminus \{0\}) : \varphi'(\pm L) = 0, [\varphi'](0) = -\alpha\varphi(0\pm)\}. \end{aligned}$$

As  $H_0^1(I)$  and  $H^1(I)$  are compactly embedded into  $L^2(I)$  both operators  $h_{\alpha,L}^D$  and  $h_{\alpha,L}^N$  have compact resolvent and their spectra consist of non-decreasing sequences of eigenvalues. The understanding of the first two eigenvalues of  $h_{\alpha,L}^D$  and  $h_{\alpha,L}^N$ , as functions of  $L$ , is important for our purposes. The next proposition is essentially proven in the paper [19].

**Proposition 3.3.** [19, Prop. 2.4, Prop. 2.5] *The following statements hold:*

(i) *there exist  $L_D = L_D(\alpha) > 0$  and  $C_D = C_D(\alpha), C'_D = C'_D(\alpha) > 0$  such that for all  $L \geq L_D$*

$$-\frac{\alpha^2}{4} < E_1(h_{\alpha,L}^D) \leq -\frac{\alpha^2}{4} + C_D e^{-C'_D L};$$

(ii) *there exist  $L_N = L_N(\alpha) > 0$  and  $C_N = C_N(\alpha), C'_N = C'_N(\alpha) > 0$  such that for all  $L \geq L_N$*

$$-\frac{\alpha^2}{4} > E_1(h_{\alpha,L}^N) \geq -\frac{\alpha^2}{4} - C_N e^{-C'_N L};$$

(iii) *for all  $L > 0$ ,  $E_2(h_{\alpha,L}^D), E_2(h_{\alpha,L}^N) \geq 0$ .*

Further, we recall a result about another family of one-dimensional Schrödinger operators. Let us introduce the interval  $J := (1, +\infty)$  and the Hilbert space  $(L^2(J), (\cdot, \cdot)_J)$ . Let  $c > 0$  be a positive constant and  $V$  be the following potential  $V(x) := x^{-2}$  on  $J$ . We consider the following closed, densely defined, symmetric, and semibounded sesquilinear forms

$$(3.2) \quad \begin{aligned} q_c^D[\varphi, \psi] &= (\varphi', \psi')_J - c(V\varphi, \psi)_J, & \text{dom } q_c^D &:= H_0^1(J), \\ q_c^N[\varphi, \psi] &= (\varphi', \psi')_J - c(V\varphi, \psi)_J, & \text{dom } q_c^N &:= H^1(J), \end{aligned}$$

in the Hilbert space  $L^2(J)$ .

By a compact perturbation argument one can show that  $\sigma_{\text{ess}}(q_c^D) = \sigma_{\text{ess}}(q_c^N) = [0, +\infty)$ . Below we provide a result on spectral asymptotics of  $q_c^N$  and  $q_c^D$ , essentially proven in [23] (cf. also [21] for further generalisations).

**Theorem 3.4.** [23, Thm. 1] *Let  $c > \frac{1}{4}$ . Then it holds that*

$$\mathcal{N}_{-E}(q_c^D) \sim \mathcal{N}_{-E}(q_c^N) \sim \frac{1}{2\pi} \sqrt{c - \frac{1}{4}} |\ln E|, \quad E \rightarrow 0+.$$

*In particular, we have  $\#\sigma_{\text{dis}}(q_c^D) = \#\sigma_{\text{dis}}(q_c^N) = \infty$ .*

**3.3. A lower bound on the counting function of  $H_{\alpha,c}$ .** In this subsection we prove Proposition 3.1.

*Proof of Proposition 3.1.* Thanks to Corollary 2.5 (i) and unitary equivalence of the forms  $Q_{\alpha,\Gamma_\theta}$  (in (2.13)) and  $Q_{\alpha,\Gamma}$  (in (2.18)) it is sufficient to prove

$$\liminf_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha,\Gamma})}{|\ln E|} \geq \frac{\cot \theta}{4\pi}.$$

We split the proof of this inequality into three steps.

**Step 1.** Let  $R > 0$ , we define the intervals  $I_1 := ((\sin \theta)^{-1} + R, +\infty)$ ,  $I_2 := (-R \tan \theta, R \tan \theta)$ , and the half-strip

$$\Pi := \{(s, t) \in \Omega_\theta : s > (\sin \theta)^{-1} + R, |t| < R \tan \theta\} = I_1 \times I_2 \subset \Omega_\theta;$$

where the  $(s, t)$ -variables and  $\Omega_\theta$  are related to the physical domain through the change of variables (2.14) (cf. Figure 3.1). We introduce the quadratic form  $Q_{\alpha, \Gamma}^\Pi$ , defined as

$$Q_{\alpha, \Gamma}^\Pi[u] := \int_\Pi (|\partial_s u|^2 + |\partial_t u|^2)(s \sin \theta + t \cos \theta) ds dt - \alpha \int_{(\sin \theta)^{-1} + R}^\infty |u(s, 0)|^2 s \sin \theta ds,$$

$$\text{dom } Q_{\alpha, \Gamma}^\Pi := \{u \in \text{dom } Q_{\alpha, \Gamma} : u = 0 \text{ on } \Omega_\theta \setminus \overline{\Pi}\}.$$

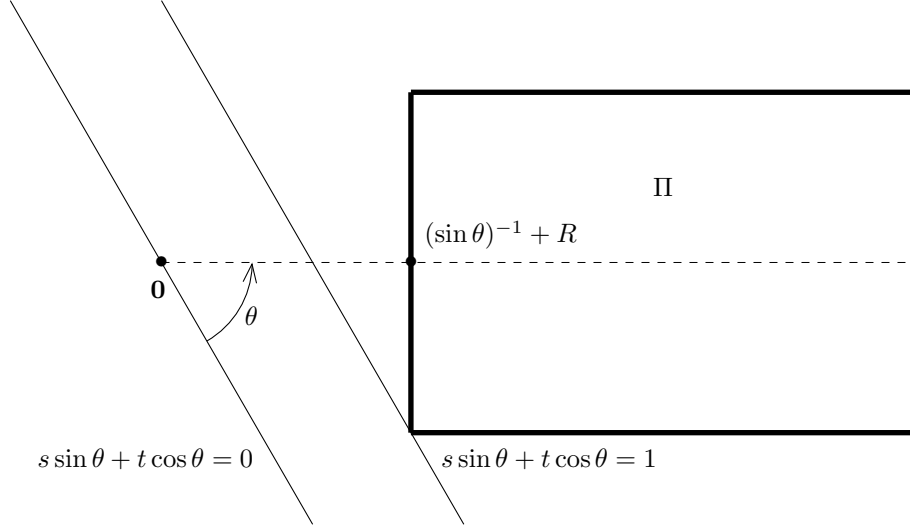


FIGURE 3.1. The inclined half-plane  $\Omega_\theta$  and the half-strip  $\Pi$ . The dashed line is the support of  $\delta$ -interaction.

Any  $u \in \text{dom } Q_{\alpha, \Gamma}^\Pi$  can be extended by zero, defining  $u_0 \in \text{dom } Q_{\alpha, \Gamma}$  such that  $Q_{\alpha, \Gamma}^\Pi[u] = Q_{\alpha, \Gamma}[u_0]$ . Then, the min-max principle yields

$$(3.3) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Gamma}^\Pi) \leq \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Gamma}).$$

**Step 2.** Let us define the unitary transform

$$\mathbf{U}: L^2(\Pi; (s \sin \theta + t \cos \theta) ds dt) \rightarrow L^2(\Pi), \quad (\mathbf{U}u)(s, t) := \sqrt{s \sin \theta + t \cos \theta} u(s, t).$$

By straightforward computation, the form  $Q_{\alpha, \Gamma}^\Pi$  is unitarily equivalent, via  $\mathbf{U}$ , to the form

$$(3.4) \quad \tilde{Q}_{\alpha, \Gamma}^\Pi[u] := \int_\Pi \left( |\partial_s u|^2 + |\partial_t u|^2 - \frac{1}{4(s \sin \theta + t \cos \theta)^2} |u|^2 \right) ds dt - \alpha \int_{(\sin \theta)^{-1} + R}^\infty |u(s, 0)|^2 ds,$$

$$\text{dom } \tilde{Q}_{\alpha, \Gamma}^\Pi := H_0^1(\Pi).$$

Next, we bound  $(s \sin \theta + t \cos \theta)^2$  from above by  $\sin^2 \theta (s + R)^2$ , obtaining

$$(3.5) \quad \tilde{Q}_{\alpha, \Gamma}^\Pi[u] \leq \int_\Pi \left( |\partial_s u|^2 + |\partial_t u|^2 - \frac{1}{4 \sin^2 \theta (s + R)^2} |u|^2 \right) ds dt - \alpha \int_{(\sin \theta)^{-1} + R}^\infty |u(s, 0)|^2 ds.$$

The right hand side of (3.5) has two blocks with separated variables. Since the Hilbert space  $L^2(\Pi)$  decomposes as  $L^2(I_1) \otimes L^2(I_2)$ , the form on the right hand side of (3.5) admits the respective representation

$$(3.6) \quad \mathfrak{q}_{1, R} \otimes \mathfrak{i}_2 + \mathfrak{i}_1 \otimes \mathfrak{q}_{2, R},$$

where  $i_k$ ,  $k = 1, 2$ , is the form of the identity operator on  $L^2(I_k)$ ; the forms  $q_{k,R}$ ,  $k = 1, 2$ , are defined in the Hilbert spaces  $L^2(I_k)$ ,  $k = 1, 2$ , as

$$\begin{aligned} q_{1,R}[\varphi, \psi] &:= (\varphi', \psi')_{I_1} - (V_R \varphi, \psi)_{I_1}, & \text{dom } q_{1,R} &:= H_0^1(I_1), \\ q_{2,R}[\varphi, \psi] &:= (\varphi', \psi')_{I_2} - \alpha \varphi(0) \overline{\psi(0)}, & \text{dom } q_{2,R} &:= H_0^1(I_2); \end{aligned}$$

where the potential  $V_R$  is given by  $V_R(s) := \frac{1}{4 \sin^2 \theta (s+R)^2}$ . Using the unitary operator

$$\mathbf{V}: L^2(1, +\infty) \rightarrow L^2(I_1), \quad (\mathbf{V}\varphi)(s) := \frac{1}{\sqrt{2R+(\sin \theta)^{-1}}} \varphi\left(\frac{s+R}{2R+(\sin \theta)^{-1}}\right),$$

one finds that the forms  $q_{1,R}$  and  $(2R+(\sin \theta)^{-1})^{-2} q_c^D$  with  $c = 1/(4 \sin^2 \theta)$  are unitarily equivalent (cf. Subsection 3.2).

Thanks to (3.3), (3.5), (3.6), and the min-max principle, we obtain

$$\begin{aligned} \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha,\Gamma}) &\geq \#\{(k, j) \in \mathbb{N}^2: E_k(q_{1,R}) + E_j(q_{2,R}) \leq -\alpha^2/4 - E\} \\ &= \sum_{j=1}^{\infty} \#\{k \in \mathbb{N}: E_k(q_{1,R}) \leq -\alpha^2/4 - E - E_j(q_{2,R})\} \\ &\geq \#\{k \in \mathbb{N}: E_k(q_{1,R}) \leq -\alpha^2/4 - E - E_1(q_{2,R})\}. \end{aligned}$$

This inequality yields

$$(3.8) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha,\Gamma}) \geq \mathcal{N}_{-\alpha^2/4-E-E_1(q_{2,R})}(q_{1,R}).$$

**Step 3.** Now, we choose  $R$  depending on the spectral parameter  $E > 0$  as follows

$$R = R(E) := M |\ln E|, \quad M > 0,$$

in particular, we have  $R(E) \rightarrow +\infty$  as  $E \rightarrow 0+$ . Let the constants  $C_D$ ,  $C'_D$  and  $L_D$  be as in Proposition 3.3 (i). Next, we choose  $M > 0$  sufficiently large such that  $C'_D M \tan \theta > 1$ . Then for  $E > 0$  sufficiently small such that  $\ln E < 0$  and  $M |\ln E| \tan \theta > L_D$ , by Proposition 3.3 (i) we have

$$|\alpha^2/4 + E_1(q_{2,R(E)})| \leq C_D \exp(C'_D M \tan \theta \ln E) = C_D E^{C'_D M \tan \theta} = o(E), \quad E \rightarrow 0+.$$

Hence,

$$(3.9) \quad f(E) := (\alpha^2/4 + E + E_1(q_{2,R(E)}))(2R(E) + (\sin \theta)^{-1})^2 = 4M^2 E |\ln E|^2 + o(E |\ln E|^2), \quad E \rightarrow 0+.$$

Using (3.8), unitary equivalence of  $q_{1,R}$  and  $(2R + (\sin \theta)^{-1})^{-2} q_c^D$  and Theorem 3.4 we get

$$\begin{aligned} \liminf_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha,\Gamma})}{|\ln(E)|} &\geq \liminf_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E-E_1(q_{2,R(E)})}(q_{1,R(E)})}{|\ln(E)|} \\ &= \liminf_{E \rightarrow 0+} \frac{\mathcal{N}_{-f(E)}(q_c^D)}{|\ln(E)|} = \frac{\cot \theta}{4\pi} \liminf_{E \rightarrow 0+} \frac{|\ln f(E)|}{|\ln E|} = \frac{\cot \theta}{4\pi}, \end{aligned}$$

where we used that  $\frac{|\ln f(E)|}{|\ln E|} \rightarrow 1$  as  $E \rightarrow 0+$  (cf. (3.9)). It ends the proof of Proposition 3.1.  $\square$

**3.4. An upper bound on the counting function of  $H_{\alpha,C}$ .** The aim of this subsection is to prove Proposition 3.2. First, we provide an auxiliary lemma, whose proof is postponed to the end of the subsection. To formulate this lemma, for  $K > 0$ , we define the domain

$$\Omega_\theta^K := \{(s, t) \in \mathbb{R}^2: s \sin \theta + t \cos \theta > 2K\} \subset \Omega_\theta,$$

and introduce the following symmetric quadratic form on the Hilbert space  $L^2(\Omega_\theta^K)$

$$(3.10) \quad \begin{aligned} Q_{\alpha,\Omega_\theta^K}[u] &:= \int_{\Omega_\theta^K} |\partial_s u|^2 + |\partial_t u|^2 - \frac{|u|^2}{4(s \sin \theta + t \cos \theta)^2} ds dt - \alpha \int_{2K(\sin \theta)^{-1}}^\infty |u(s, 0)|^2 ds, \\ \text{dom } Q_{\alpha,\Omega_\theta^K} &:= H^1(\Omega_\theta^K). \end{aligned}$$

One can check that the form  $Q_{\alpha,\Omega_\theta^K}$  is closed, densely defined and semibounded in  $L^2(\Omega_\theta^K)$ .

**Lemma 3.5.** *Let  $\alpha > 0$  and  $\theta \in (0, \pi/2)$ . For all  $K > 0$  sufficiently large, the counting functions of  $H_{\alpha, \mathcal{C}}$  and  $Q_{\alpha, \Omega_\theta^K}$  satisfy*

$$\limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(H_{\alpha, \mathcal{C}})}{|\ln E|} \leq \limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Omega_\theta^K})}{|\ln E|}.$$

Now we have all the tools to prove Proposition 3.2.

*Proof of Proposition 3.2.* According to Lemma 3.5 it is sufficient to prove that for a fixed  $K > 0$  sufficiently large, we have the bound

$$\limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Omega_\theta^K})}{|\ln E|} \leq \frac{\cot \theta}{4\pi}$$

As in the proof of Proposition 3.1, we split the proof of this inequality into three steps.

**Step 1.** Let us introduce the parameters:  $R > 0$ ,  $m := \lfloor \sqrt{R} \rfloor$ ,  $r := 2K(\sin \theta)^{-1}$  and the sequences  $r_k := 3r + kR/m$ ,  $d_k := (r_k \tan \theta)/2$  (for  $k = 0, 1, 2, \dots, m$ ). For the sake of convenience we set  $r_{m+1} = +\infty$ . We introduce the domains

$$\Lambda_k := \{(s, t) \in \mathbb{R}^2 : s \in (r_k, r_{k+1}), t \in (-d_k, d_k)\}, \quad k = 0, 1, \dots, m.$$

The inclusions  $\Lambda_k \subset \Omega_\theta^K$  hold for all  $k = 0, 1, \dots, m$ . Indeed, for any  $(s, t) \in \Lambda_k$  we have

$$s \sin \theta + t \cos \theta > r_k \sin \theta - d_k \cos \theta = \frac{r_k \sin \theta}{2} \geq \frac{3r \sin \theta}{2} = \frac{6K}{2} = 3K > 2K.$$

We also define the domains  $\Lambda_{m+1}, \Lambda_{m+2} \subset \Omega_\theta^K$  (cf. Figure 3.2) as

$$\Lambda_{m+1} := \{(s, t) \in \Omega_\theta^K : s < 6K(\sin \theta)^{-1}, |t| < K(\cos \theta)^{-1}\}, \quad \Lambda_{m+2} := \Omega_\theta^K \setminus \overline{\bigcup_{k=0}^{m+1} \Lambda_k}.$$

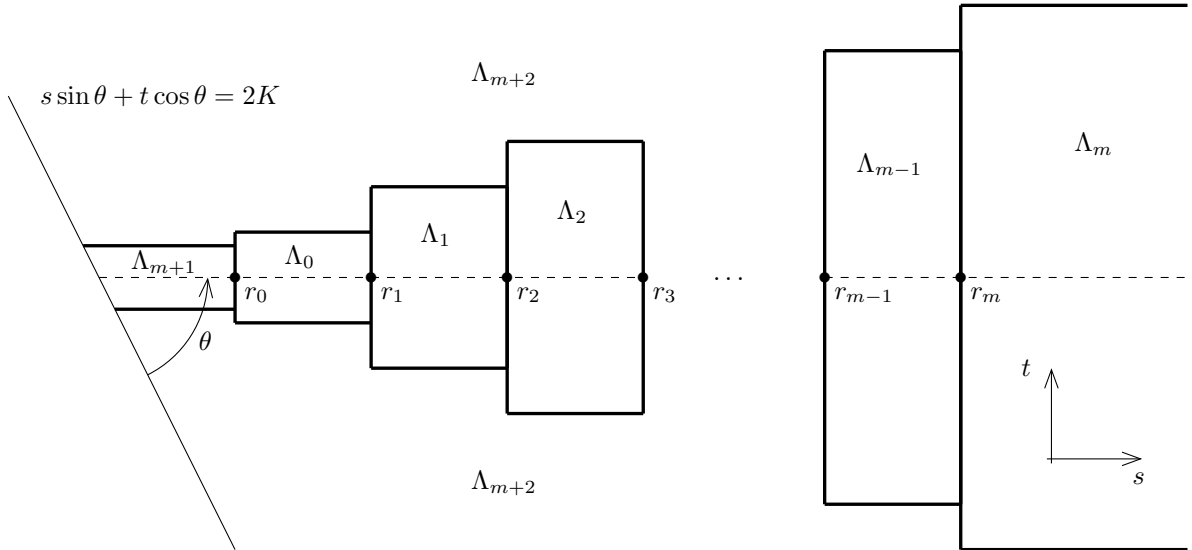


FIGURE 3.2. Sketch of the different subdomains  $\Lambda_k$  of  $\Omega_\theta^K$  ( $k \in \{0, \dots, m+2\}$ ). The dashed line is the support of the  $\delta$ -interaction.

We introduce the notation  $u_k = u|_{\Lambda_k}$  ( $k = 0, 1, \dots, (m+2)$ ) and we consider the following closed, densely defined, symmetric and semibounded quadratic form  $\tilde{Q}_{\alpha, \Omega_\theta^K}$  in  $L^2(\Omega_\theta^K)$ , defined as

$$\begin{aligned} \tilde{Q}_{\alpha, \Omega_\theta^K}[u] &:= \sum_{k=0}^{m+2} \|\nabla u_k\|_{\Lambda_k}^2 - \int_{\Omega_\theta^K} \frac{|u|^2}{4(s \sin \theta + t \cos \theta)^2} \mathrm{d}s \mathrm{d}t - \alpha \int_{2K(\sin \theta)^{-1}}^{\infty} |u(s, 0)|^2 \mathrm{d}s, \\ \mathrm{dom} \tilde{Q}_{\alpha, \Omega_\theta^K} &:= \bigoplus_{k=0}^{m+2} H^1(\Lambda_k). \end{aligned}$$

This form admits a natural decomposition into parts corresponding to the sub-domains  $\Lambda_k$

$$\tilde{Q}_{\alpha, \Omega_\theta^K}[u] = \sum_{k=0}^{m+2} Q_{\alpha, \Lambda_k}[u_k],$$

where, for  $k = 0, 1, \dots, (m+2)$ , the quadratic forms  $Q_{\alpha, \Lambda_k}$  have domains  $\mathrm{dom} Q_{\alpha, \Lambda_k} := H^1(\Lambda_k)$  and are given by

$$\begin{aligned} Q_{\alpha, \Lambda_k}[u] &:= \|\nabla u\|_{\Lambda_k}^2 - \int_{\Lambda_k} \frac{|u|^2}{4(s \sin \theta + t \cos \theta)^2} \mathrm{d}s \mathrm{d}t - \alpha \int_{r_k}^{r_{k+1}} |u(s, 0)|^2 \mathrm{d}s, \quad k = 0, \dots, m, \\ Q_{\alpha, \Lambda_{m+1}}[u] &:= \|\nabla u\|_{\Lambda_{m+1}}^2 - \int_{\Lambda_{m+1}} \frac{|u|^2}{4(s \sin \theta + t \cos \theta)^2} \mathrm{d}s \mathrm{d}t - \alpha \int_{2K(\sin \theta)^{-1}}^{r_0} |u(s, 0)|^2 \mathrm{d}s, \\ Q_{\alpha, \Lambda_{m+2}}[u] &:= \|\nabla u\|_{\Lambda_{m+2}}^2 - \int_{\Lambda_{m+2}} \frac{|u|^2}{4(s \sin \theta + t \cos \theta)^2} \mathrm{d}s \mathrm{d}t. \end{aligned}$$

As  $\mathrm{dom} Q_{\alpha, \Omega_\theta^K} \subset \mathrm{dom} \tilde{Q}_{\alpha, \Omega_\theta^K}$ , for any  $u \in \mathrm{dom} Q_{\alpha, \Omega_\theta^K}$  we have  $Q_{\alpha, \Omega_\theta^K}[u] = \tilde{Q}_{\alpha, \Omega_\theta^K}[u]$  and we get the form ordering  $\tilde{Q}_{\alpha, \Omega_\theta^K} \prec Q_{\alpha, \Omega_\theta^K}$ . The min-max principle yields, for all  $E > 0$ , the bound

$$(3.11) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Omega_\theta^K}) \leq \sum_{k=0}^{m+2} \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_k}).$$

**Step 2.** In this step we obtain bounds on the functions  $E \mapsto \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_k})$  ( $k = 0, 1, 2, \dots, (m+2)$ ). First, we bound from above the functions

$$E \mapsto \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_{m+1}}) \quad \text{and} \quad E \mapsto \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_{m+2}}).$$

Because  $H^1(\Lambda_{m+1})$  is compactly embedded into  $L^2(\Lambda_{m+1})$  the quadratic form  $Q_{\alpha, \Lambda_{m+1}}$  is associated with an operator with compact resolvent. Therefore, since the domain  $\Lambda_{m+1}$  does not depend on  $R$ , there exists a constant  $\mathfrak{c}_\theta = \mathfrak{c}_\theta(\alpha, K) > 0$ , which depends on  $\theta$ ,  $\alpha$  and  $K$  (but *not* on  $R$ ), such that, for any  $E > 0$

$$(3.12) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_{m+1}}) \leq \mathcal{N}_{-\alpha^2/4}(Q_{\alpha, \Lambda_{m+1}}) = \mathfrak{c}_\theta.$$

Further, for any  $u \in \mathrm{dom} Q_{\alpha, \Lambda_{m+2}}$ , we have

$$Q_{\alpha, \Lambda_{m+2}}[u] \geq \|\nabla u\|_{\Lambda_{m+2}}^2 - \frac{1}{16K^2} \|u\|_{\Lambda_{m+2}}^2.$$

Consequently, for a fixed  $K > 0$  such that  $\frac{1}{16K^2} < \alpha^2/4$ , the min-max principle yields  $\mathcal{N}_{-\alpha^2/4}(Q_{\alpha, \Lambda_{m+2}}) = 0$ . Hence, for any  $E > 0$ , the following equation holds

$$(3.13) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_{m+2}}) \leq \mathcal{N}_{-\alpha^2/4}(Q_{\alpha, \Lambda_{m+2}}) = 0.$$

Next, we obtain upper bounds for the functions

$$E \mapsto \sum_{k=0}^{m-1} \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_k}) \quad \text{and} \quad E \mapsto \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_m}).$$

To this end we define the non-increasing sequence

$$(3.14) \quad \varepsilon_k := \sup_{(s,t) \in \Lambda_k} \frac{1}{4(s \sin \theta + t \cos \theta)^2} = \frac{1}{r_k^2 \sin^2 \theta}, \quad k = 0, 1, \dots, m.$$

We observe that for any  $u \in \text{dom } Q_{\alpha, \Lambda_k}$

$$Q_{\alpha, \Lambda_k}[u] \geq \|\nabla u\|_{\Lambda_k}^2 - \varepsilon_k \|u\|_{\Lambda_k}^2 - \alpha \int_{r_k}^{r_{k+1}} |u(s, 0)|^2 ds, \quad k = 0, 1, \dots, m.$$

Then, we define the intervals  $I_1^k = (r_k, r_{k+1})$ ,  $I_2^k = (-d_k, d_k)$ , the potential  $V_m(s) = \frac{1}{4 \sin^2 \theta (s - r_m/2)^2}$  and the following symmetric sesquilinear forms

$$\begin{aligned} \mathfrak{q}_{1,R}^k[\varphi, \psi] &:= (\varphi', \psi')_{I_1^k} - \varepsilon_k(\varphi, \psi)_{I_1^k}, & \text{dom } \mathfrak{q}_{1,R}^k &:= H^1(I_1^k), & k = 0, 1, \dots, (m-1), \\ \mathfrak{q}_{1,R}^m[\varphi, \psi] &:= (\varphi', \psi')_{I_1^m} - (V_m \varphi, \psi)_{I_1^m}, & \text{dom } \mathfrak{q}_{1,R}^m &:= H^1(I_1^m), \\ \mathfrak{q}_{2,R}^k[\varphi, \psi] &:= (\varphi', \psi')_{I_2^k} - \alpha \varphi(0) \overline{\psi(0)}, & \text{dom } \mathfrak{q}_{2,R}^k &:= H^1(I_2^k), & k = 0, 1, \dots, m. \end{aligned}$$

One can check that all the forms are closed, densely defined and semibounded in  $L^2$ -spaces over their respective intervals. As  $L^2(\Lambda_k) = L^2(I_1^k) \otimes L^2(I_2^k)$ , we introduce the quadratic forms

$$\tilde{Q}_{\alpha, \Lambda_k} := \mathfrak{q}_{1,R}^k \otimes \mathfrak{i}_2^k + \mathfrak{i}_1^k \otimes \mathfrak{q}_{2,R}^k, \quad k = 0, 1, \dots, m,$$

and the form orderings  $\tilde{Q}_{\alpha, \Lambda_k} \prec Q_{\alpha, \Lambda_k}$  hold for all  $k = 0, 1, \dots, m$ . Here, for  $j = 1, 2$  and  $k = 0, 2, \dots, m$ ,  $\mathfrak{i}_j^k$  denote the forms of the identity operators in  $L^2(I_j^k)$ . Hence, we arrive at the bound

$$\begin{aligned} \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_k}) &\leq \mathcal{N}_{-\alpha^2/4-E}(\tilde{Q}_{\alpha, \Lambda_k}) = \#\{(l, j) \in \mathbb{N}^2 : E_l(\mathfrak{q}_{1,R}^k) + E_j(\mathfrak{q}_{2,R}^k) \leq -\alpha^2/4 - E\} \\ &= \sum_{j=1}^{\infty} \#\{l \in \mathbb{N} : E_l(\mathfrak{q}_{1,R}^k) \leq -\alpha^2/4 - E - E_j(\mathfrak{q}_{2,R}^k)\}. \end{aligned}$$

Now, we choose  $K > 0$  sufficiently large such that  $\varepsilon_1 < \alpha^2/4$ . Thanks to Proposition 3.3 (iii), we know that all the summands, for  $j > 1$ , in the above sum equal to zero. Thus, we get the same bound in a simplified form

$$(3.15) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_k}) \leq \#\{l \in \mathbb{N} : E_l(\mathfrak{q}_{1,R}^k) \leq -\alpha^2/4 - E - E_1(\mathfrak{q}_{2,R}^k)\}.$$

For  $k = 0, 1, \dots, (m-1)$  we deduce from (3.15) using (3.14) and Proposition 3.3 (ii) that for  $R > 0$  sufficiently large

$$(3.16) \quad \begin{aligned} \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_k}) &\leq \#\{l \in \mathbb{N}_0 : m^2 \pi^2 l^2 R^{-2} \leq -\alpha^2/4 + \varepsilon_k - E_1(\mathfrak{q}_{2,R}^k)\} \\ &\leq 1 + C_1 \sqrt{R} \sqrt{e^{-C_2 r_k} + r_k^{-2}} \leq 1 + \frac{C_3 \sqrt{R}}{r_k}, \end{aligned}$$

where the positive constants  $C_1, C_2$  and  $C_3$  do not depend on  $R$ . Summing the estimates (3.16) over  $k$ , we end up with

$$(3.17) \quad \begin{aligned} \sum_{k=0}^{m-1} \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_k}) &\leq m + C_3 \sqrt{R} \sum_{k=0}^{m-1} \frac{1}{r_k} \\ &\leq m + C_3 \sqrt{R} \frac{1}{r_0} + C_3 \int_0^R \frac{dx}{3r+x} \leq C_4 \sqrt{R} \end{aligned}$$

for all  $R > 0$  sufficiently large and a positive constant  $C_4$  which does not depend on  $R$ .

Further, for  $k = m$  we obtain from (3.15)

$$(3.18) \quad \begin{aligned} \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Lambda_m}) &\leq \#\{l \in \mathbb{N} : E_l(\mathfrak{q}_{1,R}^m) \leq -\alpha^2/4 - E - E_1(\mathfrak{q}_{2,R}^m)\} \\ &\leq \mathcal{N}_{-\alpha^2/4-E-E_1(\mathfrak{q}_{2,R}^m)}(\mathfrak{q}_{1,R}^m). \end{aligned}$$

Using the unitary operator

$$\mathbf{U} : L^2(1, +\infty) \rightarrow L^2(I_1^m), \quad (\mathbf{U}\psi)(s) := \sqrt{\frac{2}{r_m}} \psi\left(\frac{2s}{r_m} - 1\right),$$

one finds by direct computations that the forms  $q_{1,R}^m$  and  $\frac{4}{r_m^2}q_c^N$  with  $c = 1/(4 \sin^2 \theta)$  are unitarily equivalent; where the form  $q_c^N$  is defined in (3.2).

Combining the bound (3.11) and the estimates (3.12), (3.13), (3.17), (3.18) we obtain

$$(3.19) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha,\Omega_\theta^K}) \leq c_\theta + C_4\sqrt{R} + \mathcal{N}_{-\alpha^2/4-E-E_1(q_{2,R}^m)}(q_{1,R}^m).$$

**Step 3.** Now, we choose  $R$  depending on the spectral parameter  $E > 0$  as follows

$$R = R(E) := M|\ln E|, \quad M > 0,$$

in particular, we have  $R(E) \rightarrow +\infty$  as  $E \rightarrow 0+$ . Let the constants  $C_N$ ,  $C'_N$  and  $L_N$  be as in Proposition 3.3 (ii). Next, we choose  $M > 0$  sufficiently large such that  $C'_N M \tan \theta > 2$ . Then, for  $E > 0$  sufficiently small such that  $\ln E < 0$  and  $M|\ln E| \tan \theta/2 > L_N$ , by Proposition 3.3 (ii) we have

$$|\alpha^2/4 + E_1(q_{2,R(E)}^m)| \leq C_N \exp((C'_N \tan \theta/2)(3r + M|\ln E|)) = \tilde{C}_N E^{(C'_N M \tan \theta)/2} = o(E), \quad E \rightarrow 0+,$$

where  $\tilde{C}_N > 0$ . Hence,

$$(3.20) \quad f(E) := (r_m(E)^2/4)(\alpha^2/4 + E + E_1(q_{2,R(E)}^m)) = M^2 E |\ln E|^2/4 + o(E |\ln E|^2), \quad E \rightarrow 0+.$$

Using (3.8), unitary equivalence of  $q_{1,R}^m$  and  $\frac{4}{r_m^2}q_c^N$  and Theorem 3.4 we get

$$(3.21) \quad \begin{aligned} \limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha,\Omega_\theta^K})}{|\ln E|} &\leq \limsup_{E \rightarrow 0+} \left( \frac{c_\theta}{|\ln E|} + \frac{C_4\sqrt{M}}{\sqrt{|\ln E|}} + \frac{\mathcal{N}_{-\alpha^2/4-E-E_1(q_{2,R(E)}^m)}(q_{1,R(E)}^m)}{|\ln E|} \right) \\ &\leq \limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-f(E)}(q_c^N)}{|\ln E|} = \frac{\cot \theta}{4\pi} \limsup_{E \rightarrow 0+} \frac{|\ln f(E)|}{|\ln E|} = \frac{\cot \theta}{4\pi}. \end{aligned}$$

where we used that  $\frac{|\ln f(E)|}{|\ln E|} \rightarrow 1$  as  $E \rightarrow 0+$  (cf. (3.20)). This ends the proof of Proposition 3.2.  $\square$

Now we provide the proof of Lemma 3.5.

*Proof of Lemma 3.5.* Thanks to Corollary 2.5 (i) it is sufficient to prove that

$$\limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha,\Gamma_\theta})}{|\ln E|} \leq \limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha,\Omega_\theta^K})}{|\ln E|},$$

where the forms  $Q_{\alpha,\Gamma_\theta}$  and  $Q_{\alpha,\Omega_\theta^K}$  are defined as in (2.13) and in (3.10), respectively.

**Step 1.** Using an IMS formula we split the quadratic form of  $Q_{\alpha,\Gamma_\theta}$  into two forms, one acting on a strip-shaped geometrical domain attached to the boundary  $\partial\mathbb{R}_+^2$  of  $\mathbb{R}_+^2$ , the other one acting away from it. For this purpose, let us introduce a  $\mathcal{C}^\infty$ -smooth cut-off function  $\chi_0: \mathbb{R}_+ \rightarrow [0, 1]$  such that

$$\chi_0(r) := \begin{cases} 1, & r \leq 1, \\ 0, & r \geq 2. \end{cases}$$

We also introduce the function  $\chi_1: \mathbb{R}_+ \rightarrow [0, 1]$  such that  $\chi_0^2 + \chi_1^2 \equiv 1$ . Now, for  $K > 0$ , we define  $\chi_j^K(r) := \chi_j(K^{-1}r)$ ,  $j = 0, 1$ , and introduce the following bounded function

$$W^K(r) := |(\chi_0^K)'(r)|^2 + |(\chi_1^K)'(r)|^2 = K^{-2}(|\chi_0'(K^{-1}r)|^2 + |\chi_1'(K^{-1}r)|^2).$$

We set  $W := W^1$  (for  $K = 1$ ) and observe that  $\|W^K\|_\infty = K^{-2}\|W\|_\infty$ . Moreover, for any  $u \in \text{dom } Q_{\alpha,\Gamma_\theta}$ , a simple computation (cf. [9, Sec. 3.1]) yields

$$Q_{\alpha,\Gamma_\theta}[u] = Q_{\alpha,\Gamma_\theta}[\chi_0^K u] + Q_{\alpha,\Gamma_\theta}[\chi_1^K u] - \int_{\mathbb{R}_+^2} W^K(r)|u(r, z)|^2 r dr dz.$$

Next we introduce the sub-domains

$$\Omega_0 := \{(r, z) \in \mathbb{R}_+^2 : r \leq 2K\}, \quad \Omega_1 := \{(r, z) \in \mathbb{R}_+^2 : r \geq K\},$$



of the meridian domain  $\mathbb{R}_+^2$  (note that  $\text{supp } W^K \subset \Omega_0$ ). For  $j = 0, 1$ , we set  $\Gamma_j = \Gamma_\theta \cap \Omega_j$  and define  $I_0 := (0, 2K(\sin \theta)^{-1})$ ,  $I_1 := (K(\sin \theta)^{-1}, +\infty)$ ,  $\Sigma_0 = \{2K\} \times \mathbb{R}$ , and  $\Sigma_1 = \{K\} \times \mathbb{R}$ . Then we consider the quadratic forms  $Q_j$ ,  $j = 0, 1$ , defined as

$$(3.22) \quad \begin{aligned} Q_j[u] &:= \int_{\Omega_j} (|\partial_r u|^2 + |\partial_z u|^2 - W^K(r)|u|^2) r dr dz - \alpha \int_{I_j} |u(s \sin \theta, s \cos \theta)|^2 s \sin \theta ds, \\ \text{dom } Q_j &:= \{u|_{\Omega_j} : u \in \text{dom } Q_{\alpha, \Gamma_\theta}, u|_{\Sigma_j} = 0\}. \end{aligned}$$

Thanks to (3.22), we get that, for  $j = 0, 1$  any  $u \in \text{dom } Q_{\alpha, \Gamma_\theta}$ ,  $\chi_j^K u \in \text{dom } Q_j$  and we have the relation

$$Q_{\alpha, \Gamma_\theta}[u] = Q_0[\chi_0^K u] + Q_1[\chi_1^K u].$$

Using [10, Lem. 5.2] we find

$$(3.23) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Gamma_\theta}) \leq \mathcal{N}_{-\alpha^2/4-E}(Q_0) + \mathcal{N}_{-\alpha^2/4-E}(Q_1).$$

**Step 2.** In this step we prove that for any  $K > 0$  sufficiently large, there exists a constant  $\widehat{c}_\theta = \widehat{c}_\theta(\alpha, K) > 0$  such that

$$(3.24) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_0) \leq \widehat{c}_\theta$$

for all  $E > 0$ . First, we introduce the quadratic form  $Q'_0$ , defined as

$$Q'_0[u] := \int_{\Omega_0} (|\partial_r u|^2 + |\partial_z u|^2) r dr dz - \alpha \int_{I_0} |u(s \sin \theta, s \cos \theta)|^2 s \sin \theta ds, \quad \text{dom } Q'_0 := \text{dom } Q_0.$$

One can check that the above form is closed, symmetric, densely defined and semibounded in  $L^2(\Omega_0; r dr dz)$ . Now, for any  $u \in \text{dom } Q_0$  we have

$$(3.25) \quad Q_0[u] \geq Q'_0[u] - K^{-2} \|W\|_\infty \|u\|_{L^2(\Omega_0; r dr dz)}^2.$$

Consequently, we get that

$$(3.26) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_0) \leq \mathcal{N}_{-\alpha^2/4-E+K^{-2}\|W\|_\infty}(Q'_0).$$

Next, we choose  $K > 0$  sufficiently large such that  $-\alpha^2/4 + K^{-2}\|W\|_\infty \leq -\alpha^2/8$ . Combining (3.25) and (3.26) we obtain

$$(3.27) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_0) \leq \mathcal{N}_{-\alpha^2/8}(Q'_0).$$

Secondly, let us split the domain  $\Omega_0$  into two disjoint sub-domains (cf. Figure 3.3)

$$\Omega_{00} := \{(r, z) \in \Omega_0 : z \in (0, 2K \cot \theta)\}, \quad \Omega_{01} := \{(r, z) \in \Omega_0 : z \notin [0, 2K \cot \theta]\}.$$

For  $j = 0, 1$ , we define  $\Sigma_{0j} := \Sigma_0 \cap \partial\Omega_{0j}$  and we consider the quadratic forms  $Q'_{00}$  and  $Q'_{01}$  defined as

$$\begin{aligned} Q'_{00}[u] &:= \int_{\Omega_{00}} (|\partial_r u|^2 + |\partial_z u|^2) r dr dz - \alpha \int_{I_0} |u(s \sin \theta, s \cos \theta)|^2 s \sin \theta ds, \\ Q'_{01}[u] &:= \int_{\Omega_{01}} (|\partial_r u|^2 + |\partial_z u|^2) r dr dz, \\ \text{dom } Q'_{0j} &:= \{u : u, \partial_r u, \partial_z u \in L^2(\Omega_{0j}, r dr dz), u|_{\Sigma_{0j}} = 0\}, \quad j = 0, 1. \end{aligned}$$

One can check that the above forms are closed, symmetric, densely defined and semibounded in  $L^2(\Omega_{00}; r dr dz)$  and in  $L^2(\Omega_{01}; r dr dz)$ , respectively. For  $u \in \text{dom } Q'_0$  and  $j = 0, 1$ , we define  $u_j = u|_{\Omega_{0j}}$  and get

$$Q'_0[u] = Q'_{00}[u_0] + Q'_{01}[u_1].$$

The above equality and the min-max principle yield

$$(3.28) \quad \mathcal{N}_{-\alpha^2/8}(Q'_0) \leq \mathcal{N}_{-\alpha^2/8}(Q'_{00}) + \mathcal{N}_{-\alpha^2/8}(Q'_{01}).$$

Note that for all  $u \in \text{dom } Q'_{01}$ , we have  $Q'_{01}[u] \geq 0$ . Consequently, we get

$$(3.29) \quad \mathcal{N}_{-\alpha^2/8}(Q'_{01}) = 0.$$

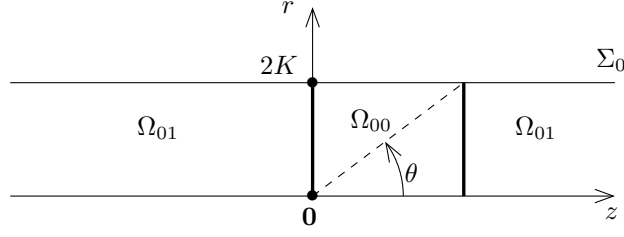


FIGURE 3.3. The domain  $\Omega_0$  and the sub-domains  $\Omega_{00}$  and  $\Omega_{01}$ . The dashed line is the support of the  $\delta$ -interaction.

Moreover, the quadratic form  $Q'_{00}$  is associated with the lowest fiber operator of a three-dimensional Schrödinger operator with a surface  $\delta$ -interaction acting on a bounded domain with mixed boundary conditions (Neumann and Dirichlet). This operator has compact resolvent and its sequence of eigenvalues goes to infinity. Hence, we obtain

$$(3.30) \quad \mathcal{N}_{-\alpha^2/8}(Q'_{00}) = \widehat{c}_\theta < \infty,$$

with some constant  $\widehat{c}_\theta = \widehat{c}_\theta(\alpha, K) > 0$ . Combining (3.27), (3.28), (3.29) and (3.30) we obtain (3.24).

**Step 3.** We remark that the domain  $\widetilde{\Omega}_1 = \{(s, t) \in \mathbb{R}^2 : s \sin \theta + t \cos \theta > K\} \subset \Omega_\theta$  is the image of  $\Omega_1$  under Rotation (2.14) and we consider the unitary transform

$$U: L^2(\Omega_1; r dr dz) \rightarrow L^2(\widetilde{\Omega}_1), \quad (Uu)(s, t) := u(s \sin \theta + t \cos \theta, s \sin \theta - t \cos \theta) \sqrt{s \sin \theta + t \cos \theta}.$$

A straightforward computation yields that the quadratic form  $Q_1$  is unitarily equivalent, via  $U$ , to the form

$$\begin{aligned} \widetilde{Q}_1[u] &:= \|\nabla u\|_{\widetilde{\Omega}_1}^2 - \int_{\widetilde{\Omega}_1} \frac{|u|^2}{4(s \sin \theta + t \cos \theta)^2} + W^K(s \sin \theta + t \cos \theta) |u|^2 ds dt - \alpha \int_{K(\sin \theta)^{-1}}^{\infty} |u(s, 0)|^2 ds \\ \text{dom } \widetilde{Q}_1 &:= H_0^1(\widetilde{\Omega}_1). \end{aligned}$$

We introduce the sub-domains of  $\widetilde{\Omega}_1$

$$\widetilde{\Omega}_{10} := \{(s, t) \in \mathbb{R}^2 : K < s \sin \theta + t \cos \theta < 2K\}, \quad \widetilde{\Omega}_{11} := \{(s, t) \in \mathbb{R}^2 : s \sin \theta + t \cos \theta > 2K\},$$

and the forms

$$\begin{aligned} \widetilde{Q}_{10}[u] &:= \|\nabla u\|_{\widetilde{\Omega}_{10}}^2 - \int_{\widetilde{\Omega}_{10}} \frac{|u|^2}{4(s \sin \theta + t \cos \theta)^2} + W^K(s \sin \theta + t \cos \theta) |u|^2 ds dt - \alpha \int_{K(\sin \theta)^{-1}}^{2K(\sin \theta)^{-1}} |u(s, 0)|^2 ds, \\ \widetilde{Q}_{11}[u] &:= \|\nabla u\|_{\widetilde{\Omega}_{11}}^2 - \int_{\widetilde{\Omega}_{11}} \frac{|u|^2}{4(s \sin \theta + t \cos \theta)^2} ds dt - \alpha \int_{2K(\sin \theta)^{-1}}^{+\infty} |u(s, 0)|^2 ds, \\ \text{dom } \widetilde{Q}_{10} &:= \{u \in H^1(\widetilde{\Omega}_{10}) : u|_{\partial \widetilde{\Omega}_{10}} = 0\}, \quad \text{dom } \widetilde{Q}_{11} := H^1(\widetilde{\Omega}_{11}). \end{aligned}$$

The above forms are closed, symmetric, densely defined and semibounded in  $L^2(\widetilde{\Omega}_{10})$  and in  $L^2(\widetilde{\Omega}_{11})$ , respectively. Once again, we get by the min-max principle

$$(3.31) \quad \mathcal{N}_{-\alpha^2/4-E}(Q_1) = \mathcal{N}_{-\alpha^2/4-E}(\widetilde{Q}_1) \leq \mathcal{N}_{-\alpha^2/4-E}(\widetilde{Q}_{10}) + \mathcal{N}_{-\alpha^2/4-E}(\widetilde{Q}_{11}).$$

**Step 4.** In this step we prove that for any  $K > 0$  sufficiently large, there exists a constant  $\widetilde{c}_\theta = \widetilde{c}_\theta(\alpha, K) > 0$  such that

$$(3.32) \quad \mathcal{N}_{-\alpha^2/4-E}(\widetilde{Q}_{10}) \leq \widetilde{c}_\theta,$$

for all  $E > 0$ . To do so, we introduce the quadratic form  $\widetilde{Q}'_{10}$  defined as

$$(3.33) \quad \widetilde{Q}'_{10}[u] := \|\nabla u\|_{\widetilde{\Omega}_{10}}^2 - \alpha \int_{K(\sin \theta)^{-1}}^{2K(\sin \theta)^{-1}} |u(s, 0)|^2 ds, \quad \text{dom } \widetilde{Q}'_{10} := \text{dom } \widetilde{Q}_{10}.$$

One can check that the above form is closed, symmetric, densely defined and semibounded in  $L^2(\tilde{\Omega}_{10})$ . Now, for any  $u \in \text{dom } \tilde{Q}_{10}$  we have

$$(3.34) \quad \tilde{Q}_{10}[u] \geq \tilde{Q}'_{10}[u] - K^{-2}(\|W\|_{\infty} + 1/4)\|u\|_{\tilde{\Omega}_{10}}^2.$$

Consequently, we get

$$\mathcal{N}_{-\alpha^2/4-E}(\tilde{Q}_{10}) \leq \mathcal{N}_{-\alpha^2/4-E+K^{-2}(\|W\|_{\infty}+1/4)}(\tilde{Q}'_{10}).$$

Next, we choose  $K > 0$  sufficiently large such that  $-\alpha^2/4 + K^{-2}(\|W\|_{\infty} + 1/4) < -\alpha^2/8$ . Combining (3.33) and (3.34) we have

$$(3.35) \quad \mathcal{N}_{-\alpha^2/4-E}(\tilde{Q}_{10}) \leq \mathcal{N}_{-\alpha^2/8}(\tilde{Q}'_{10}).$$

Then, let us split the domain  $\tilde{\Omega}_{10}$  into two disjoint sub-domains

$$\tilde{\Omega}_{100} = \{(s, t) \in \tilde{\Omega}_{10} : |t| < 1\}, \quad \tilde{\Omega}_{101} = \{(s, t) \in \tilde{\Omega}_{10} : |t| > 1\}.$$

We denote by  $\tilde{\Sigma}_0$  the image of  $\Sigma_0$  under Rotation (2.14) and, for  $j = 0, 1$ , let us define  $\tilde{\Sigma}_{0j} = \tilde{\Sigma}_0 \cap \partial\Omega_{10j}$  ( $j = 0, 1$ ). Then, we consider the quadratic forms  $\tilde{Q}'_{100}$  and  $\tilde{Q}'_{101}$  defined as

$$\begin{aligned} \tilde{Q}'_{100}[u] &:= \|\nabla u\|_{\tilde{\Omega}_{100}}^2 - \alpha \int_{K(\sin \theta)^{-1}}^{2K(\sin \theta)^{-1}} |u(s, 0)|^2 ds, \\ \tilde{Q}'_{101}[u] &:= \|\nabla u\|_{\tilde{\Omega}_{101}}^2, \\ \text{dom } \tilde{Q}'_{10j} &:= \{u \in H^1(\tilde{\Omega}_{10j}) : u|_{\tilde{\Sigma}_{0j}} = 0\}, \quad j = 0, 1. \end{aligned}$$

The above forms are closed, symmetric, densely defined and semibounded in  $L^2(\tilde{\Omega}_{100})$  and in  $L^2(\tilde{\Omega}_{101})$ , respectively. For  $u \in \text{dom } \tilde{Q}'_{10}$ , we define  $u_j = u|_{\tilde{\Omega}_{10j}}$  ( $j = 0, 1$ ) and get

$$\tilde{Q}'_{10}[u] = \tilde{Q}'_{100}[u_0] + \tilde{Q}'_{101}[u_1].$$

The above equality and the min-max principle yield

$$(3.36) \quad \mathcal{N}_{-\alpha^2/8}(\tilde{Q}'_{10}) \leq \mathcal{N}_{-\alpha^2/8}(\tilde{Q}'_{100}) + \mathcal{N}_{-\alpha^2/8}(\tilde{Q}'_{101}).$$

For all  $u \in \text{dom } \tilde{Q}'_{101}$ ,  $\tilde{Q}'_{101}[u] \geq 0$  and we get

$$(3.37) \quad \mathcal{N}_{-\alpha^2/8}(\tilde{Q}'_{101}) = 0.$$

The quadratic form  $\tilde{Q}'_{100}$  is the quadratic form of a Schrödinger operator with a  $\delta$ -interaction supported on a line segment. It acts on a bounded domain with mixed boundary conditions (Neumann and Dirichlet) thus, this operator has compact resolvent and its sequence of eigenvalues goes to infinity. Hence, we have

$$(3.38) \quad \mathcal{N}_{-\alpha^2/8}(\tilde{Q}'_{100}) = \tilde{c}_{\theta} < \infty,$$

where  $\tilde{c}_{\theta} = \tilde{c}_{\theta}(\alpha, K) > 0$ . Combining (3.35), (3.36), (3.37) and (3.38) we obtain (3.32).

**Step 5.** To conclude, inserting (3.24), (3.31) and (3.32) into (3.23) we get

$$\begin{aligned} \limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_{\alpha, \Gamma_{\theta}})}{|\ln E|} &\leq \limsup_{E \rightarrow 0+} \left( \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_0)}{|\ln E|} + \frac{\mathcal{N}_{-\alpha^2/4-E}(Q_1)}{|\ln E|} \right) \\ &\leq \limsup_{E \rightarrow 0+} \left( \frac{\mathcal{N}_{-\alpha^2/4-E}(\tilde{Q}_{11})}{|\ln E|} + \frac{\tilde{c}_{\theta}}{|\ln E|} + \frac{\tilde{c}_{\theta}}{|\ln E|} \right) = \limsup_{E \rightarrow 0+} \frac{\mathcal{N}_{-\alpha^2/4-E}(\tilde{Q}_{11})}{|\ln E|}. \end{aligned}$$

Finally, it remains to note that the form  $\tilde{Q}_{11}$  is the form  $Q_{\alpha, \Omega_{\theta}^K}$  in (3.10). This ends the proof of Lemma 3.5.  $\square$

APPENDIX A. QUADRATIC FORMS  $Q_{\alpha, \Gamma_\theta}^{[l]}$ 

The aim of this appendix is to prove the following proposition about the forms  $Q_{\alpha, \Gamma_\theta}^{[l]}$  in (2.5) and the spaces  $\mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$ ,  $\mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$  defined in Notation 2.1.

**Proposition A.1.** *Let  $d \geq 3$ ,  $l \geq 0$  and the quadratic forms  $Q_{\alpha, \Gamma_\theta}^{[l]}$  be defined as in (2.5). Then the following statements hold:*

- (i) *for  $d = 3$  and any  $l > 0$ ,  $\mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$  is a form core for  $Q_{\alpha, \Gamma_\theta}^{[l]}$ ;*
- (ii) *for  $d = 3$  and  $l = 0$  or  $d \geq 4$  and  $l \geq 0$ ,  $\mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2})$  is a form core for  $Q_{\alpha, \Gamma_\theta}^{[l]}$ .*

Before proving Proposition A.1, we need to introduce a few notation. For any function  $u \in L_{\text{cyl}}^2(\mathbb{R}^d)$  we denote by  $\tilde{u} \in L^2(\mathbb{R}^d)$  the function  $\tilde{u}(x_1, \dots, x_d) = u(r, z, \phi)$  in the physical coordinates (cf. the change of variables (2.1)). Let us fix the dimension  $d \geq 3$  and  $l \geq 0$ . We choose  $M > 0$  large enough such that for any  $\tilde{u} \in \text{dom } Q_{\alpha, \mathcal{C}}$  and any  $\hat{u} \in \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$  we have

$$Q_{\alpha, \mathcal{C}}[\tilde{u}] \geq -M \|\tilde{u}\|_{\mathbb{R}^d}^2, \quad Q_{\alpha, \Gamma_\theta}^{[l]}[\hat{u}] \geq -M \|\hat{u}\|_{L^2(\mathbb{R}_+^2; r^{d-2} dr dz)}^2.$$

We introduce the following norms associated with the quadratic forms  $Q_{\alpha, \mathcal{C}}$  and  $Q_{\alpha, \Gamma_\theta}^{[l]}$  defined, for  $\tilde{u} \in \text{dom } Q_{\alpha, \mathcal{C}}$  and  $\hat{u} \in \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$ , by

$$\|\tilde{u}\|_{Q_{\alpha, \mathcal{C}}}^2 := Q_{\alpha, \mathcal{C}}[\tilde{u}] + (M+1) \|\tilde{u}\|_{\mathbb{R}^d}^2, \quad \|\hat{u}\|_{Q_{\alpha, \Gamma_\theta}^{[l]}}^2 := Q_{\alpha, \Gamma_\theta}^{[l]}[\hat{u}] + (M+1) \|\hat{u}\|_{L^2(\mathbb{R}_+^2; r^{d-2} dr dz)}^2.$$

Now, we state three lemmas that are proven in the end of the appendix.

**Lemma A.2.** *Let  $d \geq 3$  and  $l \geq 0$ . Then the following set inclusions hold:*

- (i) *for  $d = 3$  and any  $l > 0$ ,  $\mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2}) \subset \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$ ;*
- (ii) *for  $d = 3$  and  $l = 0$  or  $d \geq 4$  and  $l \geq 0$ ,  $\mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2}) \subset \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$ .*

**Lemma A.3.** *Let  $d = 3$  and  $l > 0$ . Then the following set inclusion  $\mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2}) \cap \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]} \subset \mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$  holds.*

**Lemma A.4.** *Let  $d \geq 3$ ,  $l \geq 0$  and  $k \in \{1, \dots, c(d, l)\}$ . For any  $u(r, z, \phi) = \hat{u}(r, z) Y_{l,k}^{d-2}(\phi) \in H_{\text{cyl}}^1(\mathbb{R}^d)$  with  $\hat{u} \in L^2(\mathbb{R}_+^2; r^{d-2} dr dz)$ , there exists  $\hat{u}_n \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2}) \cap \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$  such that*

$$\|\hat{u}_n - \hat{u}\|_{Q_{\alpha, \Gamma_\theta}^{[l]}} \rightarrow 0, \quad n \rightarrow \infty.$$

Now, we have all the tools to prove Proposition A.1.

*Proof of Proposition A.1.* Let  $d \geq 3$ , fix  $l \in \mathbb{N}_0$  and  $k \in \{1, \dots, c(d, l)\}$ . Let  $\hat{u} \in \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$ , we define  $u(r, z, \phi) = \hat{u}(r, z) Y_{l,k}^{d-2}(\phi) \in L_{\text{cyl}}^2(\mathbb{R}^d)$ . One can show by direct computations that

$$\begin{aligned} \|\partial_r u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 + \|\partial_z u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 + \|r^{-1} \nabla_{\mathbb{S}^{d-2}} u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 &= \|\partial_r \hat{u}\|_{L^2(\mathbb{R}_+^2; r^{d-2} dr dz)}^2 + \|\partial_z \hat{u}\|_{L^2(\mathbb{R}_+^2; r^{d-2} dr dz)}^2 \\ &\quad + l(l+d-3) \|r^{-1} \hat{u}\|_{L^2(\mathbb{R}_+^2; r^{d-2} dr dz)}^2. \end{aligned}$$

As  $\hat{u} \in \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$  the right hand side is finite and hence  $u \in H_{\text{cyl}}^1(\mathbb{R}^d)$ . Now, thanks to Lemma A.4 we know that there exists a sequence  $\hat{u}_n \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2}) \cap \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$  such that  $\|\hat{u}_n - \hat{u}\|_{Q_{\alpha, \Gamma_\theta}^{[l]}} \rightarrow 0$  as  $n \rightarrow \infty$ . Because  $\hat{u}_n \in$

$\mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2})$  and thanks to Lemma A.2 (ii) we obtain item (ii) in Proposition A.1. To get item (i) in Proposition A.1, we use Lemma A.3, which yields  $\widehat{u}_n \in \mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$ . We conclude using Lemma A.2 (i).  $\square$

*Proof of Lemma A.2.* By definition of the form domains in (2.5), it is only necessary to check that for  $d = 3, l > 0$  and any  $\widehat{u} \in \mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$  we have  $r^{-1}\widehat{u} \in L^2(\mathbb{R}_+^2; r dr dz)$ . Let us fix  $\widehat{u} \in \mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$ , using a Taylor expansion in the  $r$ -variable, we can write  $\widehat{u}(r, z) = rg(r, z)$  with  $g \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2})$ . Consequently, we get

$$\|r^{-1}\widehat{u}\|_{L^2(\mathbb{R}_+^2; r dr dz)} = \|g\|_{L^2(\mathbb{R}_+^2; r dr dz)} < \infty,$$

which concludes the proof.  $\square$

*Proof of Lemma A.3.* Let  $d = 3$  and  $l > 0$ . We take  $\widehat{u} \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2}) \cap \text{dom } Q_{\alpha, \Gamma_\theta}^{[l]}$  and, using a Taylor expansion in the  $r$ -variable, we can write  $\widehat{u}(r, z) = \widehat{u}(0, z) + rg(r, z)$  with  $g \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2})$ . Because  $r^{-1}\widehat{u} \in L^2(\mathbb{R}_+^2; r dr dz)$ , we obtain that  $r^{-1}\widehat{u}(0, z) = r^{-1}\widehat{u}(r, z) - g(r, z) \in L^2(\mathbb{R}_+^2; r dr dz)$ . Hence, we get

$$\int_{\mathbb{R}_+^2} \frac{|\widehat{u}(0, z)|^2}{r} dr dz < \infty.$$

Finiteness of the last integral necessarily implies  $\widehat{u}(0, z) = 0$  for any  $z \in \mathbb{R}$ . Hence,  $\widehat{u}(r, z) = rg(r, z)$  and  $\widehat{u} \in \mathcal{C}_{0,0}^\infty(\overline{\mathbb{R}_+^2})$ .  $\square$

*Proof of Lemma A.4.* Let  $d \geq 3, l \in \mathbb{N}_0$  and  $k \in \{1, \dots, c(d, l)\}$ . Let the orthogonal projector  $\Pi_{l,k}$  in  $L_{\text{cyl}}^2(\mathbb{R}^d)$  be defined as in (2.3). Since  $\widetilde{u} \in H^1(\mathbb{R}^d)$ , there exists a sequence  $\widetilde{v}_n \in \mathcal{C}_0^\infty(\mathbb{R}^d)$  such that  $\|\widetilde{v}_n - \widetilde{u}\|_{H^1(\mathbb{R}^d)} \rightarrow 0$  as  $n \rightarrow \infty$ .

Further, define the modified sequence

$$u_n := \Pi_{l,k}(v_n) \in L_{\text{cyl}}^2(\mathbb{R}^d), \quad n \in \mathbb{N}.$$

We remark that

$$\widehat{u}_n(r, z) := \left\langle u_n(r, z, \cdot), Y_{l,k}^{d-2}(\cdot) \right\rangle_{\mathbb{S}^{d-2}} \in \mathcal{C}_0^\infty(\overline{\mathbb{R}_+^2}).$$

Thus, we can write  $u_n(r, z, \phi) = \widehat{u}_n(r, z) Y_{l,k}^{d-2}(\phi) \in H_{\text{cyl}}^1(\mathbb{R}^d)$ .

Next, we prove that

$$(A.1) \quad \|u_n - u\|_{H_{\text{cyl}}^1(\mathbb{R}^d)} \rightarrow 0, \quad n \rightarrow \infty.$$

By orthogonality, we have

$$\|v_n - u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 = \|u_n - u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2 + \|(I - \Pi_{l,k})v_n\|_{L_{\text{cyl}}^2(\mathbb{R}^d)}^2,$$

where the left hand side tends to zero as  $n \rightarrow \infty$ . Hence, we get

$$(A.2) \quad \|u_n - u\|_{L_{\text{cyl}}^2(\mathbb{R}^d)} \rightarrow 0, \quad n \rightarrow \infty.$$

Because  $v_n$  is smooth and compactly supported, we have the following useful commutation relations

$$\partial_r u_n = \Pi_{l,k}(\partial_r v_n) \quad \text{and} \quad \partial_z u_n = \Pi_{l,k}(\partial_z v_n),$$

which yield

$$(A.3) \quad \begin{aligned} \|\partial_r(u_n - u)\|_{L_{\text{cyl}}^2(\mathbb{R}^d)} &= \|\Pi_{l,k}(\partial_r(v_n - u))\|_{L_{\text{cyl}}^2(\mathbb{R}^d)} \leq \|\partial_r(v_n - u)\|_{L_{\text{cyl}}^2(\mathbb{R}^d)} \rightarrow 0, & n \rightarrow \infty, \\ \|\partial_z(u_n - u)\|_{L_{\text{cyl}}^2(\mathbb{R}^d)} &= \|\Pi_{l,k}(\partial_z(v_n - u))\|_{L_{\text{cyl}}^2(\mathbb{R}^d)} \leq \|\partial_z(v_n - u)\|_{L_{\text{cyl}}^2(\mathbb{R}^d)} \rightarrow 0, & n \rightarrow \infty. \end{aligned}$$

Using that spherical harmonics are eigenfunctions of the self-adjoint Laplace-Beltrami operator  $-\Delta_{\mathbb{S}^{d-2}}$  associated with the quadratic form  $H^1(\mathbb{S}^{d-2}) \ni \psi \mapsto \|\nabla_{\mathbb{S}^{d-2}} \psi\|_{\mathbb{S}^{d-2}}^2$  on  $L^2(\mathbb{S}^{d-2})$ , we have for any fixed  $(r, z) \in \mathbb{R}_+^2$

$$\begin{aligned} \langle \nabla_{\mathbb{S}^{d-2}}(u_n - u), \nabla_{\mathbb{S}^{d-2}}(v_n - u_n) \rangle_{\mathbb{S}^{d-2}} &= \left\langle (\hat{u}_n - \hat{u})(-\Delta_{\mathbb{S}^{d-2}} Y_{l,k}^{d-2}), (I - \Pi_{l,k})v_n \right\rangle_{\mathbb{S}^{d-2}} \\ &= l(l + d - 3) \langle u_n - u, (I - \Pi_{l,k})v_n \rangle_{\mathbb{S}^{d-2}} \\ &= l(l + d - 3) \langle \Pi_{l,k}(v_n - u_n), (I - \Pi_{l,k})v_n \rangle_{\mathbb{S}^{d-2}} = 0. \end{aligned}$$

The above relation gives

$$\|\nabla_{\mathbb{S}^{d-2}}(v_n - u)\|_{\mathbb{S}^{d-2}}^2 = \|\nabla_{\mathbb{S}^{d-2}}(u_n - u)\|_{\mathbb{S}^{d-2}}^2 + \|\nabla_{\mathbb{S}^{d-2}}(v_n - u_n)\|_{\mathbb{S}^{d-2}}^2.$$

Multiplying the latter equality by  $r^{-2}$  and integrating in  $(r, z)$  we get

$$(A.4) \quad \|r^{-1} \nabla_{\mathbb{S}^{d-2}}(u_n - u)\|_{L^2_{\text{cyl}}(\mathbb{R}^d)}^2 \leq \|r^{-1} \nabla_{\mathbb{S}^{d-2}}(v_n - u)\|_{L^2_{\text{cyl}}(\mathbb{R}^d)}^2 \leq \|v_n - u\|_{H^1_{\text{cyl}}(\mathbb{R}^d)}^2 \rightarrow 0, \quad n \rightarrow \infty.$$

Finally, combining (A.2), (A.3), (A.4) and the fact that  $u \in H^1_{\text{cyl}}(\mathbb{R}^d)$  we get (A.1).

According to [2, Prop. 3.1 and its proof] and Notation 2.1, the  $H^1_{\text{cyl}}(\mathbb{R}^d)$ -norm is equivalent to the norm  $\|\cdot\|_{Q_{\alpha,c}}$  after suitable identification of functions. Hence, (A.1) yields

$$\|\tilde{u}_n - \tilde{u}\|_{Q_{\alpha,c}} \rightarrow 0, \quad n \rightarrow \infty.$$

By direct computation, we get

$$\|\tilde{u}_n - \tilde{u}\|_{Q_{\alpha,c}} = \|\hat{u}_n - \hat{u}\|_{Q_{\alpha,\Gamma_\theta}^{[l]}},$$

which concludes the proof of Lemma A.4.  $\square$

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